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DIVERGENCE-FREE AND CURL-FREE WAVELETS ON THE SQUARE FOR NUMERICAL SIMULATIONS

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Abstract

We present a construction of divergence-free and curl-free wavelets on the square, that could satisfy suitable boundary conditions. This construction is based on the existence of biorthogonal multiresolution analyses (BMRA) on $[0, 1]$, linked by differentiation and integration. We introduce new BMRAs and wavelets for the spaces of divergence-free and curl-free vector functions on the square. The interest of such constructions is illustrated on examples including the Helmholtz-Hodge decomposition of vector flows and the Stokes problem.

1 Introduction

In many physical problems, like the numerical simulation of incompressible flows or in electromagnetism, the solution has to fulfill a divergence-free condition. For the numerical treatment of the relevant equations (Navier-Stokes equation in fluid mechanism or Maxwell's equation in electromagnetism) it is helpful to have at hand bases satisfying a divergence-free or a curl-free condition. In the context of solution schemes for Partial Differential Equations, wavelet bases provide very efficient algorithms, characterized by a reduced computational complexity, with respect to standard methods[7]. Divergence-free wavelet bases on $\mathbb{R}^d$, with compact support, were originally defined by Lemarié-Rieusset in 1992[20] and applied by Urban to the numerical solution of the Stokes-problem[25]. In the periodic case, anisotropic divergence-free wavelets have been constructed in[15], and firstly used to compute the numerical solution of the Navier-Stokes equation in velocity-pressure formulation[14]. Such numerical scheme requires, at each time-step, the Helmholtz-Hodge decomposition of the nonlinear term, which is no more divergence-free. In Fourier space, this decomposition writes explicitly, whereas in wavelet domain, it can be computed using divergence-free and curl-free wavelets [26, 13]. In the general case with physical boundary conditions, it is the key of Navier-Stokes numerical simulations to have at hand an explicit and efficient procedure to compute the Helmholtz-Hodge decomposition of the nonlinear term.

Precisely, the Helmholtz-Hodge decomposition of a vector field $\mathbf{u}$ on the square $\Omega = [0, 1]^2$ consists in splitting $\mathbf{u}$ into a divergence-free part and a curl-free part[16]. A first

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formulation leading to an orthogonal splitting is the following: there exist a stream function ψ and a potential field q such that:

$$\begin{cases} \mathbf{u} = \mathbf{curl}\ \psi + \nabla q & \text{in} \quad \Omega \\ \mathbf{curl}\ \psi \cdot \vec{\nu} = 0 & \text{on} \quad \Gamma = \partial\Omega \end{cases} \tag{1}$$

where $\vec{\nu}$ is the outward normal to Γ. This decomposition corresponds to the orthogonal splitting of the space $(L^2(\Omega))^2$:

$$(L^2(\Omega))^2 = \mathcal{H}_{div,\Gamma}(\Omega) \oplus \mathcal{H}_{div}^{\perp}(\Omega) \tag{2}$$

where

$$\mathcal{H}_{div,\Gamma}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 : \mathbf{div}(\mathbf{u}) = 0,\ \mathbf{u} \cdot \vec{\nu}|_{\Gamma} = 0\}$$

is the divergence-free function space with velocity tangent to the boundary. It can also be seen as the "curl" space:

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} = \mathbf{curl}\ \psi\ ;\ \psi \in H_0^1(\Omega)\}$$

On the other side the space of gradient functions

$$\mathcal{H}_{div}^{\perp}(\Omega) = \{\nabla q\ ;\ q \in H^1(\Omega)\}$$

corresponds to a curl-free function space[16].

Other types of boundary conditions for the divergence-free space can be considered. For instance, the decomposition:

$$(L^2(\Omega))^2 = \mathcal{H}_{div}(\Omega) \oplus \{\nabla q\ ;\ q \in H_0^1(\Omega)\} \tag{3}$$

where now

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 : \mathbf{div}(\mathbf{u}) = 0\} = \{\mathbf{u} = \mathbf{curl}\ \psi\ ;\ \psi \in H^1(\Omega)\}$$

does not incorporate boundary condition on Γ. In fluid mechanism, this type of boundary condition is less considered since in general Γ corresponds to a physical wall that cannot be crossed by fluid particles (except for porous media). A more useful condition corresponds to the homogeneous Dirichlet boundary condition on Γ, which leads to:

$$(H_0^1(\Omega))^2 = \mathcal{H}_{div,0}(\Omega) \oplus \mathcal{H}_{div,0}^{\perp}(\Omega) \tag{4}$$

where now

$$\mathcal{H}_{div,0}(\Omega) = \mathcal{H}_{div}(\Omega) \cap (H_0^1(\Omega))^2$$

while $\mathcal{H}_{div,0}^{\perp}(\Omega)$ is a subspace of $\mathcal{H}_{div}^{\perp}(\Omega) \cap (H_0^1(\Omega))^2$, see[16] for details. For sake of simplicity, we will focus in this article to the divergence-free spaces being involved in decompositions (3,4).

Accordingly, the objective of the present paper is to provide multiresolution analyses and wavelet bases of the spaces $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{div}^{\perp}(\Omega)$. We present in the next section a new construction, based on wavelets on the interval $[0,1]$ that should satisfy homogeneous boundary conditions, like in [21]. The key idea of our construction is based on a couple of wavelet bases on the interval, linked by differentiation, like in the theoretical approach of Jouini-Lemarié-Rieusset[18]. The construction of divergence-free and curl-free approximation spaces and wavelets satisfying suitable boundary conditions are then straightforward. Moreover, our method extends readily to the cube $[0.1]^d$ by tensor product [22].

The outline of the paper is as follows. Section 2 details the principles of the construction of divergence-free and curl-free BMRAs and wavelets on the square. Section 3 is dedicated to the description of the divergence-free fast wavelet transform. Finally, section 4 presents two examples of use of these new wavelets in numerical simulations: the Helmholtz decompositions and the Stokes problem.

2 Divergence-free and curl-free wavelets on the square

Divergence-free wavelets on the whole space $\mathbb{R}^d$ have been firstly constructed by Battle-Federbush [2] in the orthogonal case. Since these previous functions don't have compact support they where not implemented, contrarily to the biorthogonal bases arising from the construction proposed by Lemarié-Rieusset in [20]. Urban was the first who used them in a practical problem, the Stokes problem [25]. Later, Urban proposed to extend this construction to derive curl-free wavelets [26]. An alternative fast decomposition into divergence-free wavelets was proposed by Deriaz-Perrier, based on anisotropic (tensor-product) wavelets in the periodic case [15]. The objective below is to extend these constructions to the square $[0,1]^2$, following the construction principle already exposed in the theoretical work of Jouini-Lemarié-Rieusset [18].

2.1 Construction principle

The construction of divergence-free wavelets on the cube $[0,1]^d$ uses the same arguments as in the whole domain $\mathbb{R}^d$ [20, 18]. The key ingredient is to have at hand two one-dimensional multiresolution analyses (V_j^1) and (V_j^0) of $L^2(0,1)$ linked by differentiation:

$$\frac{d}{dx} V_j^1 \;=\; V_j^0 \tag{5}$$

On the interval $[0,1]$, following [18], the biorthogonal spaces should satisfy:

$$\tilde{V}_j^0 \;=\; H_0^1(0,1) \cap \int_0^x \tilde{V}_j^1 = \Big\{ f \;:\; f' \in \tilde{V}_j^1 \text{ and } \; f(0) = f(1) = 0 \Big\} \tag{6}$$

The existence of such spaces follows from the fundamental proposition of Lemarié-Rieusset, used at the begining to construct divergence-free wavelets on the whole space $\mathbb{R}^d$ [20]:

Proposition 2.1
Let $(V_j^1(\mathbb{R}))$ be a multiresolution analysis (MRA) of $L^2(\mathbb{R})$, with differentiable and compactly supported scaling function φ^1 and associated wavelet ψ^1. Then there exists a MRA $(V_j^0(\mathbb{R}))$, with associated scaling function φ^0 and wavelet ψ^0, such that:

$$(\varphi^1)'(x) \;=\; \varphi^0(x) \;-\; \varphi^0(x-1) \qquad \text{and} \qquad (\psi^1)'(x) \;=\; 4\,\psi^0(x) \tag{7}$$

Similar relations hold for the dual functions $\left(\tilde{\varphi}^1, \tilde{\psi}^1\right)$ and $\left(\tilde{\varphi}^0, \tilde{\psi}^0\right)$ of the primal ones (φ^1, ψ^1) and (φ^0, ψ^0):

$$\int_x^{x+1} \tilde{\varphi}^1(t)\, dt = \tilde{\varphi}^0(x) \qquad \text{and} \qquad (\tilde{\psi}^0)'(x) \;=\; -4\,\tilde{\psi}^1(x) \tag{8}$$

In [18], Jouini-Lemarié-Rieusset prove that from the scaling functions $(\varphi^1, \tilde{\varphi}^1)$, $(\varphi^0, \tilde{\varphi}^0)$ and wavelets $(\psi^1, \tilde{\psi}^1)$, $(\psi^0, \tilde{\psi}^0)$ of proposition 2.1, it is possible to exhibit multiresolution analyses on the interval satisfying the relations (5,6).

Our objective in the present paper is first to provide an effective construction of such multiresolution analyses, which enable boundary conditions, fast wavelet algorithms, approximation results and practical computations. This is done in section 2.2.1.

Then the construction of biorthogonal MRAs and wavelets bases of $\mathcal{H}_{div}(\Omega)$ (with suitable boundary conditions) and $\mathcal{H}_{div}^{\perp}(\Omega)$, are obtained by considering resp. the **curl** of $(V_j^1 \otimes V_j^1)$, and the **grad** of $(V_j^1 \otimes V_j^1)$. This will be described in section 2.3.

2.2 Construction of biorthogonal MRAs on $[0,1]$ linked by differentiation

We detail now the construction of spaces $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ satisfying (5) and (6). We proceed in two steps. The first step (construction of $(V_j^1, \tilde{V}_j^1)$, section 2.2.1) is classical and based on biorthogonal multiresolution analyses on the interval reproducing polynomials [9, 11, 21, 17, 4], but it is needed to introduce the sekels. The second step (section 2.2.2) introduces a practical and new way to provide spaces $(V_j^0, \tilde{V}_j^0)$ and associated wavelets. It is based on proposition 2.1.

2.2.1 Construction of $(V_j^1, \tilde{V}_j^1)$ on $[0,1]$ with polynomial reproduction $(r, \tilde{r})$

We first recall the definition of a biorthogonal multiresolution analysis (BMRA) on $[0,1]$ [18, 8]:

Definition 2.1
The sequence $(V_j^1, \tilde{V}_j^1)$, $j \geq j_{min}$ ($j_{min} \in \mathbb{N}^*$) is a biorthogonal MRA of approximation order $(r, \tilde{r})$ on the interval $[0,1]$ associated to the generators $(\varphi^1, \tilde{\varphi}^1)$, if it satisfies:
(i) $V_j^1 \subset V_{j+1}^1$, $\tilde{V}_j^1 \subset \tilde{V}_{j+1}^1$ and $\overline{\cup_{j\geq j_{min}} V_j^1} = \overline{\cup_{j\geq j_{min}} \tilde{V}_j^1} = L^2(0,1)$.
(ii) $V_j^{1,int} \subset V_j^1 \subset V_j^1(\mathbb{R})|_{[0,1]}$, $\tilde{V}_j^{1,int} \subset \tilde{V}_j^1 \subset \tilde{V}_j^1(\mathbb{R})|_{[0,1]}$.
(iii) V_j^1 and $\tilde{V}_j^1$ are finite dimensional biorthogonal spaces spanned by biorthogonal bases $\{\varphi_{j,k}^1 : \ k \in \Delta_j\}$ and $\{\tilde{\varphi}_{j,k}^1 : \ k \in \Delta_j\}$: $< \varphi_{j,k}^1, \tilde{\varphi}_{j,k'}^1 >= \delta_{k,k'}, \ \forall \ k, k' \in \Delta_j$.
(iv) V_j^1 and $\tilde{V}_j^1$ have respectively r and $\tilde{r}$ polynomial exactness.

In point (ii), $V_j^1(\mathbb{R})|_{[0,1]}$ means the restriction of $V_j^1(\mathbb{R})$-functions to the interval $[0,1]$, whereas $V_j^{1,int}$ means interior functions of $V_j^1(\mathbb{R})$, as introduced below (definitions 2.2,2.4), and same for the biorthogonal spaces. The dimension $\Delta_j \approx 2^j$ of spaces V_j^1 and $\tilde{V}_j^1$ will be explicited later.

To construct such spaces $(V_j^1, \tilde{V}_j^1)$, as described in the numerous and now classical approaches [9, 11, 21, 17, 4], we start with generators $(\varphi^1, \tilde{\varphi}^1)$, that are biorthogonal scaling functions of a BMRA on $\mathbb{R}$. We suppose that φ^1 is compactly supported on $[n_{min}, n_{max}]$ and reproduces polynomials up to degree $r-1$:

$$0 \leq \ell \leq r-1, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \tilde{p}_\ell^1(k) \ \varphi^1(x-k) \tag{9}$$

with $\tilde{p}^1_\ell(k) = \langle \frac{x^\ell}{\ell!}, \tilde{\varphi}^1(x-k)\rangle$. We suppose also that $\tilde{\varphi}^1$ reproduces polynomials up to degree $\tilde{r}-1$:

$$0 \le \ell \le \tilde{r}-1, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} p^1_\ell(k)\ \tilde{\varphi}^1(x-k) \tag{10}$$

with $p^1_\ell(k) = \langle \frac{x^\ell}{\ell!}, \varphi^1(x-k)\rangle$.

To define a BMRA on $[0,1]$ we first define the set of *interior scaling functions*:

Definition 2.2
Let $\delta_\flat,\ \delta_\sharp \in \mathbb{N}$ be two fixed parameters. For $j \ge 0$, interior scaling functions of V^1_j are defined as scaling functions $\varphi^1_{j,k}(x) = 2^{j/2}\varphi^1(2^j x - k)$ whose supports are included into $[\frac{\delta_\flat}{2^j}, 1-\frac{\delta_\sharp}{2^j}] \subset [0,1]$.
If supp $\varphi^1 = [n_{min}, n_{max}]$, they correspond to indices k such that:

$$\delta_\flat - n_{min} \le k \le 2^j - \delta_\sharp - n_{max}$$

The space generated by interior scaling functions is then given by:

$$V^{1,int}_j = span\{\varphi^1_{j,k}\ ;\ k = k_\flat,\ 2^j - k_\sharp\}$$

with $k_\flat = \delta_\flat - n_{min}$ and $k_\sharp = \delta_\sharp + n_{max}$.

Similarly, let $\tilde{\delta}_\flat,\ \tilde{\delta}_\sharp \in \mathbb{N}$ be two parameters. Interior scaling functions of $\tilde{V}^1_j$ are defined as scaling functions $\tilde{\varphi}^1_{j,k}(x) = 2^{j/2}\tilde{\varphi}^1(2^j x - k)$ whose supports are included into $[\frac{\tilde{\delta}_\flat}{2^j}, 1-\frac{\tilde{\delta}_\sharp}{2^j}]$. The space generated by interior scaling functions is then given by:

$$\tilde{V}^{1,int}_j = span\{\tilde{\varphi}^1_{j,k}\ ;\ k = \tilde{k}_\flat,\ 2^j - \tilde{k}_\sharp\}$$

with $\tilde{k}_\flat = \tilde{\delta}_\flat - \tilde{n}_{min}$ and $\tilde{k}_\sharp = \tilde{\delta}_\sharp + \tilde{n}_{max}$, if the support of $\tilde{\varphi}^1$ is $[\tilde{n}_{min}, \tilde{n}_{max}]$.

Remark 1
The parameters $(\delta_\flat,\ \delta_\sharp,\ \tilde{\delta}_\flat,\ \tilde{\delta}_\sharp)$ are "free" parameters (chosen as small as possible), and chosen in practice to adjust the dimension of the spaces V^1_j and $\tilde{V}^1_j$.

To preserve the polynomial reproduction (9, 10) on the interval $[0,1]$, we follow the approach of [21, 4] and define edge scaling functions at the edge 0:

Definition 2.3
The edge scaling functions at the edge 0 are defined by:

$$0 \le \ell \le r-1, \quad \Phi^{1,\flat}_\ell(x) = \sum_{k=1-n_{max}}^{k_\flat - 1} \tilde{p}^1_\ell(k)\ \varphi^1_k(x)\ \chi_{[0,+\infty[}$$

and the duals:

$$0 \le \ell \le \tilde{r}-1, \quad \tilde{\Phi}^{1,\flat}_\ell(x) = \sum_{k=1-\tilde{n}_{max}}^{\tilde{k}_\flat - 1} p^1_\ell(k)\ \tilde{\varphi}^1_k(x)\ \chi_{[0,+\infty[}$$

At the edge 1, the edge scaling functions $\Phi^{1,\sharp}_{j,\ell}$ are constructed on $]-\infty,1]$ by symmetry, using the transform $Tf(x)=f(1-x)$.

As usual, one define the multiresolution spaces V^1_j on $[0,1]$, by the direct sum:

$$V^1_j = V^{1,\flat}_j \oplus V^{1,int}_j \oplus V^{1,\sharp}_j \tag{11}$$

where:

$$\begin{aligned} V^{1,\flat}_j &= span\{2^{j/2}\Phi^{1,\flat}_\ell(2^j x)\ ;\ \ell=0,\cdots,r-1\} \\ V^{1,\sharp}_j &= span\{2^{j/2}\Phi^{1,\sharp}_\ell(2^j(1-x))\ ;\ \ell=0,\cdots,r-1\} \end{aligned}$$

are the *edge spaces*. In practice we have to choose $j \geq j_{min}$ where j_{min} is the smallest integer which verifies:

$$j_{min} > \log_2[n_{max}-n_{min}+\delta_\sharp+\delta_\flat]$$

This condition ensures that the supports of edge scaling functions at 0 do not intersect the supports of edge scaling functions at 1.
The polynomial reproduction in V^1_j is then satisfied since, for $0 \leq \ell \leq r-1$ and $x \in [0,1]$ we have:

$$\frac{2^{j/2}(2^j x)^\ell}{\ell!} = 2^{j/2}\Phi^{1,\flat}_\ell(2^j x) + \sum_{k=k_\flat}^{2^j-k_\sharp} \tilde{p}^1_\ell(k)\ \varphi^1_{j,k}(x) + 2^{j/2}\Phi^{1,\sharp}_\ell(2^j(1-x)) \tag{12}$$

Similarly, multiresolution spaces $\tilde{V}^1_j$ are defined with the same structure, allowing the polynomial reproduction up to degree $\tilde{r}-1$:

$$\tilde{V}^1_j = span\{\tilde{\Phi}^{1,\flat}_{j,\ell}\}_{\ell=0,\tilde{r}-1} \oplus \tilde{V}^{1,int}_j \oplus span\{\tilde{\Phi}^{1,\sharp}_{j,\ell}\}_{\ell=0,\tilde{r}-1} \tag{13}$$

In order to obtain the equality between dimensions of V^1_j and $\tilde{V}^1_j$, we have to adjust the parameters $\tilde{\delta}_\flat = \tilde{k}_\flat - \tilde{n}_{max}$ and $\tilde{\delta}_\sharp = \tilde{k}_\sharp + \tilde{n}_{min}$ in the definition 2.2 such that:

$$k_\flat - r = \tilde{k}_\flat - \tilde{r} \qquad \text{and} \qquad k_\sharp - r = \tilde{k}_\sharp - \tilde{r} \tag{14}$$

We get:

$$\dim(V^1_j) = \dim(\tilde{V}^1_j) = 2^j - (\delta_\flat+\delta_\sharp) - (n_{max}-n_{min}) + 2r + 1$$

where $(\delta_\flat,\ \delta_\sharp)$ remain "free" parameters of the construction. Like for V^1_j we have to choose $j \geq \tilde{j}_{min}$ with $\tilde{j}_{min} > \log_2[\tilde{n}_{max}-\tilde{n}_{min}+\tilde{\delta}_\sharp+\tilde{\delta}_\flat]$.

The last point of the construction lies in the biorthogonalization process of the new basis functions, since edge scaling functions of V^1_j and $\tilde{V}^1_j$ are not biorthogonal. Several biorthogonalization methods exist [1, 11, 17, 21], here we apply on one hand the method proposed by Dahmen and al. [11] when using B-spline generators, and on the other hand a Gram-Schmidt process with Daubechies orthogonal generators [21]. In both cases, it requires the inversion of the Gram matrix associated with the two systems, which for orthogonal and B-Spline generators is non singular [11, 21].
Finally, the spaces $(V^1_j, \tilde{V}^1_j)_{j\geq\max\{j_{min},\tilde{j}_{min}\}}$, constitute a biorthogonal MRA of $L^2(0,1)$ in the sense of definition 2.1.

Moreover, homogeneous boundary conditions can be simply imposed to V_j^1, of the form $f^{(\lambda)}(\alpha) = 0$ at point $\alpha = 0$ or 1, with $\lambda = 0, \cdots, r-1$, by removing the edge scaling function $\Phi_\lambda^{1,\flat}$ if $\alpha = 0$ or $\Phi_\lambda^{1,\sharp}$ if $\alpha = 1$ in the definition 2.3 of edge spaces (see [21] for more details). In such case, we also remove the function $\tilde{\Phi}_\lambda^{1,\flat}$ or $\tilde{\Phi}_\lambda^{1,\sharp}$ from $\tilde{V}_j^1$ prior to biorthogonalization.

2.2.2 Construction of $(V_j^0, \tilde{V}_j^0)$ on $[0,1]$ linked by differentiation /integration with $(V_j^1, \tilde{V}_j^1)$

We will now construct spaces $(V_j^0, \tilde{V}_j^0)$, related to the spaces $(V_j^1, \tilde{V}_j^1)$ of section 2.2.1 by the relations (5,6) of differentiation/integration.
Given $(\varphi^1, \tilde{\varphi}^1)$ biorthogonal scaling functions with approximation orders $(r, \tilde{r})$ $(r > 1)$, and compact supports $[n_{min}, n_{max}]$, $[\tilde{n}_{min}, \tilde{n}_{max}]$, we consider $(\varphi^0, \tilde{\varphi}^0)$ arising from proposition 2.1. Then $(\varphi^0, \tilde{\varphi}^0)$ satisfy some properties on $\mathbb{R}$, that we recall below.

First, the scaling functions $(\varphi^0, \tilde{\varphi}^0)$ are defined such that:

$$\frac{d}{dx}\varphi^1(x) = \varphi^0(x) - \varphi^0(x-1) \quad \text{and} \quad \int_x^{x+1} \tilde{\varphi}^1(t)dt = \tilde{\varphi}^0(x) \tag{15}$$

This implies that φ^0 has for compact support $[n_{min}, n_{max}-1]$, and reproduces polynomials up to degree $r-2$:

$$0 \leq \ell \leq r-2, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \tilde{p}_\ell^0(k)\ \varphi^0(x-k) \tag{16}$$

with $\tilde{p}_\ell^0(k) = \langle \frac{x^\ell}{\ell!}, \tilde{\varphi}^0(x-k)\rangle$.
Equation (8) implies: $\tilde{p}_\ell^0(k) = \tilde{p}_{\ell+1}^1(k) - \tilde{p}_{\ell+1}^1(k-1)$ for $\ell = 0, \cdots, r-2$.

In the same way $\tilde{\varphi}^0$ has for compact support $[\tilde{n}_{min}-1, \tilde{n}_{max}]$, and reproduces polynomials up to degree $\tilde{r}$:

$$0 \leq \ell \leq \tilde{r}, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} p_\ell^0(k)\ \tilde{\varphi}^0(x-k) \tag{17}$$

with $p_\ell^0(k) = \langle \frac{x^\ell}{\ell!}, \varphi^0(x-k)\rangle$.
Equation (7) implies: $p_\ell^1(k) = p_{\ell+1}^0(k+1) - p_{\ell+1}^0(k)$ for $\ell = 1, \cdots, \tilde{r}$.

Like for V_j^1, we first define the set of *interior scaling functions* of V_j^0:

Definition 2.4
Let $\delta_\flat,\ \delta_\sharp \in \mathbb{N}$ and $k_\flat = \tilde{\delta}_\flat - \tilde{n}_{min}$ and $k_\sharp = \tilde{\delta}_\sharp + \tilde{n}_{max}$ be the parameters introduced in definition 2.2. For $j \geq 0$, the interior scaling functions of V_j^0 are defined as scaling functions $\varphi_{j,k}^0(x) = 2^{j/2}\varphi^0(2^jx-k)$ whose supports are included into $[\frac{\delta_\flat}{2^j}, 1-\frac{\delta_\sharp}{2^j}] \subset [0,1]$. Since supp $\varphi^0 = [n_{min}, n_{max}-1]$, the space generated by interior scaling functions is given by:

$$V_j^{0,int} = span\{\varphi_{j,k}^0\ ;\ k = k_\flat,\ 2^j - k_\sharp + 1\}$$

Similarly, let $\tilde\delta_\flat$, $\tilde\delta_\sharp \in \mathbb{N}$, and $\tilde k_\flat = \tilde\delta_\flat - \tilde n_{min}$, $\tilde k_\sharp = \tilde\delta_\sharp + \tilde n_{max}$ be fixed by definition 2.2 and relation (14). Interior scaling functions of $\tilde V_j^0$ are defined as scaling functions $\tilde\varphi^0_{j,k}(x) = 2^{j/2}\varphi^0(2^jx-k)$ whose supports are included into $[\frac{\tilde\delta_\flat}{2^j}, 1-\frac{\tilde\delta_\sharp}{2^j}]$. The space generated by interior scaling functions is then given by:

$$\tilde V_j^{0,int} = span\{\tilde\varphi^0_{j,k} \; ; \; k = \tilde k_\flat + 1, \; 2^j - \tilde k_\sharp\}$$

To preserve the polynomial reproduction (16,17) on $[0,1]$ in $(V_j^0, \tilde V_j^0)$, we define *edge scaling functions* at the edge 0:

Definition 2.5

The edge scaling functions at the edge 0 are defined by:

$$0 \le \ell \le r-2, \qquad \Phi_\ell^{0,\flat}(x) = \sum_{k=2-n_{max}}^{k_\flat - 1} \tilde p^0_\ell(k) \; \varphi^0_k(x) \; \chi_{[0,+\infty[}$$

and the biorthogonal ones, vanishing at 0:

$$1 \le \ell \le \tilde r, \qquad \tilde\Phi_\ell^{0,\flat}(x) = \sum_{k=1-\tilde n_{max}}^{\tilde k_\flat} p^0_\ell(k) \; \tilde\varphi^0_k(x) \; \chi_{[0,+\infty[}$$

At the edge 1, the *edge scaling functions* $\Phi^{0,\sharp}_{j,\ell}$ and $\tilde\Phi^{0,\sharp}_{j,\ell}$ are constructed by symmetry, using the transform $Tf(x) = f(1-x)$.

Remark 2

Following Jouini-Lemarié-Rieusset [18], to preserve the commutation between the derivation and the multiscale projectors, the space $\tilde V_j^0$ should satisfy (6): $\tilde V_j^0 \subset H_0^1(0,1)$. Indeed we impose by construction homogeneous Dirichlet boundary conditions to $\tilde V_j^0$, since we do not consider $\tilde\Phi_0^{0,\flat}$ and $\tilde\Phi_0^{0,\sharp}$ ($\ell = 0$) in definition 2.5.

The multiresolution spaces V_j^0 on $[0,1]$ are then defined by:

$$V_j^0 = V_j^{0,\flat} \oplus V_j^{0,int} \oplus V_j^{0,\sharp}$$

where:

$$V_j^{0,\flat} = span\{2^{j/2}\Phi_\ell^{0,\flat}(2^jx) \; ; \; \ell = 0, \cdots, r-2\}$$
$$V_j^{0,\sharp} = span\{2^{j/2}\Phi_\ell^{0,\sharp}(2^j(1-x)) \; ; \; \ell = 0, \cdots, r-2\}$$

The polynomial reproduction up to degree $r-2$ in V_j^0 is then ensured. In practice $j > j_{min}$ where the parameter j_{min} is now adapted to both BMRA $(V_j^1, \tilde V_j^1)$ and $(V_j^0, \tilde V_j^0)$ by:

$$j_{min} > \max\{\log_2[n_{max} - n_{min} + \delta_\sharp + \delta_\flat + 1], \log_2[\tilde n_{max} - \tilde n_{min} + \tilde\delta_\sharp + \tilde\delta_\flat + 1]\}$$

Multiresolution spaces $\tilde V_j^0$ are defined with similar structure, allowing a polynomial reproduction up to degree $\tilde r$, and satisfying vanishing boundary conditions at 0 and 1.

$$\tilde V_j^0 = span\{\tilde\Phi^{0\flat}_{j,\ell}\}_{\ell=1,\tilde r} \oplus \tilde V_j^{0,int} \oplus span\{\tilde\Phi^{1,\sharp}_{j,\ell}\}_{\ell=1,\tilde r} \tag{18}$$

A simple calculation shows that:

$$\dim(V_j^0) = 2^j - k_\sharp - k_\flat + 2r \quad and \quad \dim(\tilde V_j^0) = 2^j - \tilde k_\sharp - \tilde k_\flat + 2\tilde r \tag{19}$$

Since the parameters $k_\flat$, $k_\sharp$, $\tilde k_\flat$ and $\tilde k_\sharp$ are chosen to satisfy equation (14), we obtain: $\dim(V_j^0) = \dim(\tilde V_j^0)$ The following proposition proves that $\frac{d}{dx} V_j^1 = V_j^0$ and $\frac{d}{dx}\tilde V_j^0 \subset \tilde V_j^1$.

Proposition 2.2
(i) The interior scaling functions of (V_j^1, V_j^0) *and* $(\tilde{V}_j^1, \tilde{V}_j^0)$ *introduced in definitions 2.2, 2.4 satisfy:*

$$\frac{d}{dx}(\varphi_{j,k}^1) = 2^j[\varphi_{j,k}^0 - \varphi_{j,k+1}^0] \qquad\qquad \frac{d}{dx}(\tilde{\varphi}_{j,k}^0) = 2^j[\tilde{\varphi}_{j,k-1}^1 - \tilde{\varphi}_{j,k}^1]$$

and $\dim(V_j^{0,int}) = \dim(V_j^{1,int}) + 1$, $\dim(\tilde{V}_j^{0,int}) = \dim(\tilde{V}_j^{1,int}) - 1$.
(ii) The edge scaling functions of (V_j^1, V_j^0) *introduced in definitions 2.3, 2.5 satisfy: for* $\ell = 1, r-1$,

$$\begin{array}{ll} (\Phi_0^{1,\flat})'(x) = -\varphi_{k_\flat}^0, & (\Phi_\ell^{1,\flat})'(x) = \Phi_{\ell-1}^{0,\flat} - \tilde{p}_\ell^1(k_\flat - 1)\ \varphi_{k_\flat}^0 \\ (\Phi_0^{1,\sharp})'(1-x) = \varphi_{2-k_\sharp}^0, & (\Phi_\ell^{1,\sharp})'(1-x) = -\Phi_{\ell-1}^{0,\sharp}(1-x) + \tilde{p}_\ell(2-k_\sharp)\ \varphi_{2-k_\sharp}^0 \end{array}$$

whereas those of $(\tilde{V}_j^1, \tilde{V}_j^0)$ *are linked by: for* $\ell = 1, \tilde{r}-1$,

$$\begin{array}{ll} (\tilde{\Phi}_0^{0,\flat})' = -\tilde{\varphi}_{\tilde{k}_\flat}^1, & (\tilde{\Phi}_\ell^{0,\flat})' = \tilde{\Phi}_{\ell-1}^{1,\flat} - p_\ell^0(\tilde{k}_\flat)\ \tilde{\varphi}_{\tilde{k}_\flat}^1 \\ (\tilde{\Phi}_0^{0,\sharp}(1-x))' = \tilde{\varphi}_{1-\tilde{k}_\sharp}^1, & (\tilde{\Phi}_\ell^{0,\sharp}(1-x))' = -\tilde{\Phi}_{\ell-1}^{1,\sharp}(1-x) + p_\ell^0(2-\tilde{k}_\sharp)\ \tilde{\varphi}_{1-\tilde{k}_\sharp}^1 \end{array}$$

Proof 2.1 *The point (i) follows from (15) and since interior functions are defined by* $\varphi_{j,k}^1(x) = 2^{j/2}\varphi(2^jx - k)$ *(and the same for* $\varphi_{j,k}^0, \tilde{\varphi}_{j,k}^1, \tilde{\varphi}_{j,k}^0$*).*
The point (ii) comes directly from definitions 2.3, 2.5 of edge scaling functions. We focus on the first line of equalities: at edge 0 *and for* $0 \le \ell \le r-1$*, the edge scaling functions of* (V_j^1) *are defined by dilation of:*

$$\Phi_\ell^{1,\flat}(x) = \sum_{k=1-n_{max}}^{k_\flat - 1} \tilde{p}_\ell^1(k)\ \varphi_k^1(x)\chi_{[0,+\infty[} \quad \textit{with} \quad \varphi_k^1(x) = \varphi^1(x-k)$$

Differentiating in $]0,+\infty[$*, one obtains for* $\ell = 0$ *(*$\tilde{p}_0^1(k) = 1, \forall k$*):*

$$(\Phi_0^{1,\flat})' = \sum_{k=1-n_{max}}^{k_\flat - 1} (\varphi_k^0 - \varphi_{k+1}^0)\ \chi_{]0,+\infty[} = \varphi_{1-n_{max}}^0\ \chi_{]0,+\infty[} - \varphi_{k_\flat}^0 = -\varphi_{k_\flat}^0$$

since supp $\varphi_{1-n_{max}}^0 = [n_{min} - n_{max} + 1, 0]$.
In the same way, for $\ell = 1, r-1$*:*

$$\begin{aligned} (\Phi_\ell^{1,\flat})' &= \sum_{k=1-n_{max}}^{k_\flat - 1} \tilde{p}_\ell^1(k)\ (\varphi_k^0 - \varphi_{k+1}^0)\ \chi_{]0,+\infty[} \\ &= \sum_{k=2-n_{max}}^{k_\flat - 1} [\tilde{p}_\ell^1(k) - \tilde{p}_\ell^1(k-1)]\ \varphi_k^0\chi_{[0,+\infty[} - p_\ell^1(k_\flat - 1)\ \varphi_{k_\flat}^0 \end{aligned}$$

From (8), since $\tilde{p}_{\ell-1}^0(k) = \tilde{p}_\ell^1(k) - \tilde{p}_\ell^1(k-1)$ *we get:*

$$(\Phi_\ell^{1,\flat})' = \sum_{k=2-n_{max}}^{k_\flat - 1} \tilde{p}_{\ell-1}^0(k)\ \varphi_k^0\chi_{[0,+\infty[} - \tilde{p}_\ell^1(k_\flat - 1)\ \varphi_{k_\flat}^0 = \Phi_{\ell-1}^{0,\flat} - \tilde{p}_\ell^1(k_\flat - 1)\ \varphi_{k_\flat}^0$$

This proves the relation between edge scaling functions at 0 *of* V_j^1 *and* V_j^0*. The proof for edge scaling functions at edge* 1 *and in the biorthogonal spaces* $\tilde{V}_j^1$ *and* $\tilde{V}_j^0$ *is obtained with similar arguments.*

For easy reading, the two pairs of biorthogonal bases of $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ will be denoted by $(\varphi_{j,k}^1, \tilde{\varphi}_{j,k}^1)$ and $(\varphi_{j,k}^0, \tilde{\varphi}_{j,k}^0)$ respectively. The oblique projector on V_j^1 parallel to $(\tilde{V}_j^1)^\perp$ will be denoted by $\mathcal{P}_j^1$:

$$\mathcal{P}_j^1 : L^2(0,1) \to V_j^1, \quad f \mapsto \mathcal{P}_j^1(f) = \sum_k \langle f, \tilde{\varphi}_{j,k}^1 \rangle \ \varphi_{j,k}^1 \tag{20}$$

while $\tilde{\mathcal{P}}_j^1$ will denote its adjoint, and $\mathcal{P}_j^0$, $\tilde{\mathcal{P}}_j^0$ the biorthogonal projectors associated with $(V_j^0, \tilde{V}_j^0)$.

The following proposition proves that the constructed biorthogonal MRAs take place in the theoretical framework of Jouini-Lemarié-Rieusset in [18].

Proposition 2.3
The two pairs of biorthogonal spaces $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ are related to:

$$\frac{d}{dx} V_j^1 = V_j^0 \quad \textit{and} \quad \tilde{V}_j^0 = H_0^1 \cap \int_0^x \tilde{V}_j^1, \quad \textit{with} \quad \tilde{V}_j^0 \subset H_0^1(0,1)$$

Proof 2.2
The inclusions $\frac{d}{dx} V_j^1 \subset V_j^0$ and $\frac{d}{dx} \tilde{V}_j^0 \subset \tilde{V}_j^1$ are straightforward according to proposition 2.2. Moreover the equality of dimensions between spaces ends the proof.

We then define the change of bases between the spaces $(\frac{d}{dx} V_j^1, \frac{d}{dx} \tilde{V}_j^0)$ and $(V_j^0, \tilde{V}_j^1)$ as follows.

Definition 2.6
Let (L_j^1, L_j^0) and $(\tilde{L}_j^1, \tilde{L}_j^0)$ be the two pairs of sparse matrices defined by the change of bases between spaces involved in proposition 2.3 :

$$\frac{d}{dx} \varphi_{j,k}^1 = \sum_n (L_j^1)_{k,n} \ \varphi_{j,n}^0, \qquad \frac{d}{dx} \tilde{\varphi}_{j,k}^0 = \sum_n (\tilde{L}_j^0)_{k,n} \ \tilde{\varphi}_{j,n}^1 \tag{21}$$

and

$$-\int_0^x \varphi_{j,k}^0 = \sum_m (L_j^0)_{k,m} \ \varphi_{j,m}^1, \qquad -\int_0^x \tilde{\varphi}_{j,k}^1 = \sum_m (\tilde{L}_j^1)_{k,m} \ \tilde{\varphi}_{j,m}^0 \tag{22}$$

Remark 3
The matrices (L_j^1, L_j^0) and $(\tilde{L}_j^1, \tilde{L}_j^0)$ are rectangular and from the biorthogonality of spaces $(V_j^0, \tilde{V}_j^0)$, it comes:

$$L_j^0 (\tilde{L}_j^0)^T = I_{\dim(V_j^0)}$$

where I denotes the matrix identity. Except for the first scaling function $\Phi_{j,0}^{1,\flat}$, which does not satisfy homogeneous Dirichlet boundary condition at 0, similarly we have:

$$L_j^1 (\tilde{L}_j^1)^T = I_{\dim(V_j^1)-1}$$

In addition, the definition of $\tilde{L}_j^1$ (22) must include the scaling functions $\tilde{\Phi}_{j,0}^{0,\flat}$ and $\tilde{\Phi}_{j,0}^{0,\sharp}$, if not, this definition leads to $\int_0^1 \tilde{\varphi}_{j,m}^1 = 0$, which is not true.

With this definition, we prove the commutation between multiscale projectors and differentiation.

Proposition 2.4
Let $(\mathcal{P}_j^1, \tilde{\mathcal{P}}_j^1)$ and $(\mathcal{P}_j^0, \tilde{\mathcal{P}}_j^0)$ be the multiscale projectors defined by (20):

(i) $\forall\ f \in H^1(0,1),\ \ \frac{d}{dx} \circ \mathcal{P}_j^1 f = \mathcal{P}_j^0 \circ \frac{d}{dx} f.$

(ii) $\forall\ f \in H_0^1(0,1),\ \ \frac{d}{dx} \circ \tilde{\mathcal{P}}_j^0 f = \tilde{\mathcal{P}}_j^1 \circ \frac{d}{dx} f.$

Proof 2.3 *(i) The relation of commutation $\frac{d}{dx} \circ \mathcal{P}_j^1 f = \mathcal{P}_j^0 \circ \frac{d}{dx} f$ was demonstrated first in [18] in the general setting. We will now prove the commutation of the projectors with derivation (ii). Let $(\varphi_{j,k}^0, \tilde{\varphi}_{j,k}^0)$ be a pair of biorthogonal scaling functions of $(V_j^0, \tilde{V}_j^0)$ and as $\tilde{\varphi}_{j,k}^0 \in H_0^1(0,1)$, from proposition 2.2 we obtain:*

$$-\langle \frac{d}{dx}\tilde{\varphi}_{j,k}^0, \int_0^x \varphi_{j,k'}^0 \rangle = \langle \tilde{\varphi}_{j,k}^0, \varphi_{j,k'}^0 \rangle = \delta_{k,k'} = \sum_n (\tilde{L}_j^0)_{k,n} (L_j^0)_{k',n} = [\tilde{L}_j^0 L_j^{0T}]_{k,k'}$$

where L_j^0 and $\tilde{L}_j^0$ are introduced in (21) and (22). For $f \in H_0^1(0,1)$, since $\langle f, \varphi_{j,k}^0 \rangle = -\langle \frac{d}{dx} f, \int_0^x \varphi_{j,k}^0 \rangle$ we get:

$$\begin{aligned} \frac{d}{dx}\tilde{\mathcal{P}}_j^0(f) &= \sum_k \langle f, \varphi_{j,k}^0 \rangle \frac{d}{dx}\tilde{\varphi}_{j,k}^0 = \sum_n \sum_m \sum_k (L_j^0)_{k,m} (\tilde{L}_j^0)_{k,n} \langle \frac{d}{dx} f, \varphi_{j,m}^1 \rangle\ \tilde{\varphi}_{j,n}^1 \\ &= \sum_n \sum_m \delta_{n-m} \langle \frac{d}{dx} f, \varphi_{j,m}^1 \rangle\ \tilde{\varphi}_{j,n}^1 = \sum_n \langle \frac{d}{dx} f, \varphi_{j,n}^1 \rangle\ \tilde{\varphi}_{j,n}^1 = \tilde{\mathcal{P}}_j^1(\frac{d}{dx} f) \end{aligned}$$

This proves the relation (ii).

2.2.3 Wavelet spaces

We begin with the construction of wavelet bases of the biorthogonal MRA $(V_j^1, \tilde{V}_j^1)$. This point is classical, although different kinds of wavelets may be designed [1, 9, 11, 21, 17, 4]. For $j \geq j_{min}$, the biorthogonal wavelet spaces associated to V_j^1 and $\tilde{V}_j^1$ are in all cases defined by:

$$W_j^1 = V_{j+1}^1 \cap (\tilde{V}_j^1)^\perp \quad \text{and} \quad \tilde{W}_j^1 = \tilde{V}_{j+1}^1 \cap (V_j^1)^\perp$$

The wavelet space W_j^1 has the following structure:

$$W_j^1 = W_j^{1,\flat} \oplus W_j^{1,int} \oplus W_j^{1,\sharp}$$

where

$$\left\{ \begin{array}{lcl} W_j^{1,\flat} &=& span\{2^{j/2}\Psi_\ell^{1,\flat}(2^j x)\ ;\ \ell = 0,\ p_\flat - 1\} \\ W_j^{1,int} &=& span\{\psi_{j,k}^1\ ;\ k = p_\flat,\ 2^j - p_\sharp - 1\} \\ W_j^{1,\sharp} &=& span\{2^{j/2}\Psi_\ell^{1,\sharp}(2^j(1-x))\ ;\ \ell = 0,\ p_\sharp - 1\} \end{array} \right. \tag{23}$$

$p_\flat$ and $p_\sharp$ introduced above are suitable integers to ensure that the support of each *interior wavelet* $\psi_{j,k}^1(x) = 2^{j/2}\psi^1(2^j x - k)$ of $W_j^{1,int}$ is included into $[\frac{\delta_\flat}{2^j}, 1 - \frac{\delta_\sharp}{2^J}]$. Recall that the support of ψ^1 (wavelet on $\mathbb{R}$) is $[\frac{n_{min} - \tilde{n}_{max} + 1}{2}, \frac{n_{max} - \tilde{n}_{min} + 1}{2}]$, then we deduce:

$$p_\flat = \lfloor \frac{\tilde{n}_{max} + k_\flat - 1}{2} \rfloor \quad \text{and} \quad p_\sharp = \lfloor \frac{k_\sharp - \tilde{n}_{min} + 1}{2} \rfloor$$

To construct the *edge wavelets* $\Psi_{j,\ell}^{1,\flat}$ and $\Psi_{j,\ell}^{1,\sharp}$ we have followed the work of Grivet-Talocia and Tabacco [17]. The biorthogonal spaces $\tilde{W}_j^1$ have the same structure and the wavelet bases of the two spaces must to be biorthogonalized identically as the scaling functions. The resulting wavelet bases are denoted by $\{\psi_{j,k}^1\}_{k=0,2^j-1}$ and $\{\tilde{\psi}_{j,k}^1\}_{k=0,2^j-1}$ without distinction.

Another advantage of this construction is the existence of a fast wavelet transforms because the scaling functions and wavelets satisfy both a two-scale relation. Indeed, there are sparse matrices H_j^1, $\tilde{H}_j^1$, G_j^1 and $\tilde{G}_j^1$ such that:

$$\varphi_{j,k}^1 = \sum_n (H_j^1)_{k,n}\varphi_{j+1,n}^1 \qquad \text{and} \qquad \psi_{j,k}^1 = \sum_n (G_j^1)_{k,n}\varphi_{j+1,n}^1$$

$$\tilde{\varphi}_{j,k}^1 = \sum_n (\tilde{H}_j^1)_{k,n}\tilde{\varphi}_{j+1,n}^1 \qquad \text{and} \qquad \tilde{\psi}_{j,k}^1 = \sum_n (\tilde{G}_j^1)_{k,n}\tilde{\varphi}_{j+1,n}^1$$

The main objective of this section is now to construct biorthogonal wavelet bases of W_j^0 and $\tilde{W}_j^0$, that will be linked to $\psi_{j,k}^1$ and $\tilde{\psi}_{j,k}^1$ by differentiation/integration. A first result in this direction is given by the following proposition:

Proposition 2.5
Let $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ be two biorthogonal multiresolution analyses satisfying proposition 2.3. The spaces W_j^0 and $\tilde{W}_j^0$ defined by:

$$W_j^0 = \frac{d}{dx}W_j^1 \quad \text{and} \quad \tilde{W}_j^0 = \int_0^x \tilde{W}_j^1 \tag{24}$$

correspond to biorthogonal wavelet spaces associated to $(V_j^0, \tilde{V}_j^0)$ in the classical sense:

$$W_j^0 = V_{j+1}^0 \cap (\tilde{V}_j^0)^\perp \qquad \text{and} \qquad \tilde{W}_j^0 = \tilde{V}_{j+1}^0 \cap (V_j^0)^\perp$$

Proof 2.4 *First we prove the relation: $\frac{d}{dx}W_j^1 = V_{j+1}^0 \cap (\tilde{V}_j^0)^\perp$. Let $w_j^1 \in W_j^1$, we get from proposition 2.4 (i):*

$$\mathcal{P}_j^0(\frac{d}{dx}w_j^1) = \frac{d}{dx}\mathcal{P}_j^1(w_j^1) = 0$$

thus $\frac{d}{dx}w_j^1 \in (\tilde{V}_j^0)^\perp$. Moreover, following the proposition 2.3 (i):

$$\frac{d}{dx}w_j^1 \in \frac{d}{dx}V_{j+1}^1 = V_{j+1}^0$$

This implies $\frac{d}{dx}W_j^1 \subset V_{j+1}^0$ and thus $\frac{d}{dx}W_j^1 \in V_{j+1}^0 \cap (\tilde{V}_j^0)^\perp$. The equality is obtained by the equality between both space dimension.
To prove the relation between $\tilde{W}_j^0$ and $\tilde{W}_j^1$, as $\tilde{W}_j^0 \subset H_0^1(0,1)$, one can remark that:

$$\tilde{W}_j^0 = \int_0^x \tilde{W}_j^1 \Leftrightarrow \frac{d}{dx}\tilde{W}_j^0 = \tilde{W}_j^1$$

The relation $\frac{d}{dx}\tilde{W}_j^0 = \tilde{W}_j^1$ is then proved as before.

As suggested in [18], the wavelet bases of W_j^0 and $\tilde{W}_j^0$ are constructed directly by differentiating and integrating the wavelets of W_j^1 and $\tilde{W}_j^1$:

Definition 2.7
Let $\{\psi^1_{j,k}\}_{k=0,2^j-1}$ and $\{\tilde\psi^1_{j,k}\}_{k=0,2^j-1}$ be two biorthogonal wavelet bases of W^1_j and $\tilde W^1_j$ respectively. The wavelets of W^0_j and $\tilde W^0_j$ are defined by:

$$\psi^0_{j,k} = 2^{-j}(\psi^1_{j,k})' \quad \textit{and} \quad \tilde\psi^0_{j,k} = -2^j \int_0^x \tilde\psi^1_{j,k}$$

Interior wavelets $\psi^0_{j,k}(x) = 2^{j/2}\psi^0(2^jx-k)$ in this definition correspond to the classical wavelets arising from previous constructions [9, 11, 21, 17, 4], ψ^0 being a wavelet on $\mathbb{R}$ associated to the derivative φ^0. On the other hand, in standard constructions, the edge wavelets do not verify the relations:

$$\frac{d}{dx}\Psi^{1,\flat}_{j,k} = 2^j\Psi^{0,\flat}_{j,k} \quad \text{or} \quad \tilde\Psi^{0,\flat}_{j,k} = -2^j\int_0^x \tilde\Psi^{1,\flat}_{j,k} \tag{25}$$

The following proposition guarantees (25) and the new edge wavelets provided by definition 2.7 preserve fast algorithms since they satisfy a two-scale relationship.

Proposition 2.6
Let $\{\psi^1_{j,k}\}_{k=0,2^j-1}$ and $\{\tilde\psi^1_{j,k}\}_{k=0,2^j-1}$ be two biorthogonal wavelet bases of W^1_j and $\tilde W^1_j$ associated respectively to filters G^1_j and $\tilde G^1_j$:

$$\psi^1_{j,k} = \sum_n (G^1_j)_{k,n}\ \varphi^1_{j+1,n} \quad \textit{and} \quad \tilde\psi^1_{j,k} = \sum_n (\tilde G^1_j)_{k,n}\ \tilde\varphi^1_{j+1,n}$$

Then the following propositions hold:
(i) The system $\{\psi^0_{j,k} = 2^{-j}(\psi^1_{j,k})'\}_{k=0,2^j-1}$ and $\{\tilde\psi^0_{j,k} = -2^j\int_0^x \tilde\psi^1_{j,k}\}_{k=0,2^j-1}$ form biorthogonal wavelet bases of W^0_j and $\tilde W^0_j$ respectively.
(ii) There exist sparse matrices G^0_j and $\tilde G^0_j$ defined by:

$$G^0_j = 2^{-j}G^1_jL^1_{j+1} \quad \textit{and} \quad \tilde G^0_j = -2^j\tilde G^1_jL^{0T}_{j+1} \tag{26}$$

and the wavelets $\psi^0_{j,k}$ and $\tilde\psi^0_{j,k}$ satisfy:

$$\psi^0_{j,k} = \sum_n (G^0_j)_{k,n}\varphi^0_{j+1,n} \quad \textit{and} \quad \tilde\psi^0_{j,k} = \sum_n (\tilde G^0_j)_{k,n}\tilde\varphi^0_{j+1,n}$$

Proof 2.5 (i) *The construction of biorthogonal bases by differentiation /integration is an idea of Cieselski-Figiel [5] used by Jouini-Lemarié-Rieusset [18]. Moreover, from proposition 2.5 we can see that the two systems are independent and they generate the corresponding spaces.*
(ii) To derive the filters, we only use the definition of wavelets. So, we have:

$$\begin{aligned} 2^j\psi^0_{j,k} &= \sum_n (G^1_j)_{k,n}(\varphi^1_{j+1,n})' = \sum_{n,m}(G^1_j)_{k,n}(L^1_{j+1})_{n,m}\varphi^0_{j+1,m} \\ &= \sum_m [G^1_jL^1_{j+1}]_{k,m}\varphi^0_{j+1,m} = 2^j\sum_m (G^0_j)_{k,m}\varphi^0_{j+1,m} \end{aligned}$$

which gives the filter of $\psi^0_{j,k}$. Similarly, for $\tilde\psi^0_{j,k}$ we obtain:

$$\begin{aligned} \frac{d}{dx}\tilde\psi^0_{j,k} &= \sum_m (\tilde G^0_j)_{k,m}(\tilde\varphi^0_{j+1,m})' = \sum_{m,n}(\tilde G^0_j)_{k,m}(\tilde L^0_{j+1})_{m,n}\tilde\varphi^1_{j+1,n} \\ &= \sum_n [\tilde G^0_j\tilde L^0_{j+1}]_{k,n}\tilde\varphi^1_{j+1,n} = -2^j\sum_n (\tilde G^1_j)_{k,n}\tilde\varphi^1_{j+1,n} \end{aligned}$$

Taking into account the relation $[\tilde{L}_j^0 L_j^{0T}]_{k,k'} = \delta_{k,k'}$, *there are summary:*

$$G_j^0 = 2^{-j} G_j^1 L_{j+1}^1 \quad \textit{and} \quad \tilde{G}_j^0 = -2^j \tilde{G}_j^1 L_{j+1}^{0T}$$

This completes the proof.

Remark 4
The above construction of wavelets $\psi_{j,k}^0$ *and* $\tilde{\psi}_{j,k}^0$ *has two main interests: their filters are directly accessible from those of* $\psi_{j,k}^1$ *and* $\tilde{\psi}_{j,k}^1$ *and there is no need for biorthogonalization as for classical constructions.*

Example 1
To illustrate, we give the plot of edge scaling functions and wavelets at 0 *in* $(V_j^1, \tilde{V}_j^1)$ *on Figure 1. The generators* $(\varphi^1, \tilde{\varphi}^1)$ *used are biorthogonal B-Spline with* $r = \tilde{r} = 3$. *Then we have:* $n_{min} = -1$, $n_{max} = 2$, $\tilde{n}_{min} = -3$ *and* $\tilde{n}_{max} = 4$. *The "free" integer parameters are chosen as* $\delta_{min} = \delta_{max} = 2$ *and* $\tilde{\delta}_{min} = \tilde{\delta}_{max} = 0$. *On Figure 2, we plot the corresponding edge scaling functions and wavelets of* $(V_j^0, \tilde{V}_j^0)$ *at* 0. *The Figure 3 and Figure 4 show the non zeros elements of filters* G_j^0 *and* $\tilde{G}_j^0$ *and matrices* L_j^0 *and* L_j^1 *respectively, for* $j = 6$.

2.3 Biorthogonal MRA of $\mathcal{H}_{div}(\Omega)$

Let Ω be the square $[0,1]^2$. The aim of the present section is to provide a divergence-free MRA and wavelet bases of the space $\mathcal{H}_{div}(\Omega)$ [16]:

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 : \ \mathbf{div}(\mathbf{u}) = 0 \text{ and } \mathbf{u} \cdot \vec{\nu} = 0\}$$

Since this space is equal to:

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} = \mathbf{curl}\ \chi \ ; \ \chi \in H_0^1(\Omega)\}$$

our construction consists in taking the curl of a regular MRA of the two-dimensional scalar space $H_0^1(\Omega)$.

Such MRA of $H_0^1(\Omega)$ is usually defined as tensor-product of one-dimensional MRA of $H_0^1(0,1)$. We now consider a regular one-dimensional MRA satisfying homogeneous boundary conditions:

$$V_j^D = V_j^1 \cap H_0^1(0,1)$$

as constructed in section 2.2, and which takes the form:

$$V_j^D = span\{\Phi_{j,\ell}^{1,\flat} \ ; \ \ell = 1, r-1\} \oplus V_j^{1,int} \oplus span\{\Phi_{j,\ell}^{1,\sharp} \ ; \ \ell = 1, r-1\}$$

To simplify, we denote by $\varphi_{j,k}^D$ the scaling functions of V_j^D:

$$V_j^D = span\{\varphi_{j,k}^D \ ; \ k = 0, \ 2^j - k_\sharp - k_\flat + 2r - 2\}$$

and $\psi_{j,k}^D$ the corresponding wavelets. With these notations, the divergence-free scaling function spaces are defined below.

Definition 2.8
For $j \geq j_{min}$, *the divergence-free scaling function spaces* $\mathbf{V}_j^{div}$ *are defined by:*

$$\mathbf{V}_j^{div} = \mathbf{curl}(V_j^D \otimes V_j^D) = span\{\Phi_{j,\mathbf{k}}^{div}\} \tag{27}$$

where the divergence-free scaling functions are given by:

$$\Phi^{div}_{j,\mathbf{k}} := \mathbf{curl}[\varphi^D_{j,k_1} \otimes \varphi^D_{j,k_2}],\ \ j \geq j_{min} \tag{28}$$

The spaces $\mathbf{V}^{div}_j$ defined above constitute an increasing sequence of subspaces of $(L^2(\Omega))^2$:

$$\mathbf{V}^{div}_j \subset \mathbf{V}^{div}_{j+1}$$

of dimension:

$$\begin{aligned}\dim(\mathbf{V}^{div}_j) = \dim(V^D_j)^2 &= (2^j - k_\sharp - k_\flat + 2r - 1)^2 \\ &= \big(2^j - (n_{max} - n_{min}) - (\delta_\flat + \delta_\sharp) + 2r - 1\big)^2\end{aligned}$$

We will also consider a more standard multiresolution analysis $\vec{\mathbf{V}}_j$ of $(L^2(\Omega))^2$ defined as:

$$\vec{\mathbf{V}}_j = (V^1_j \otimes V^0_j) \times (V^0_j \otimes V^1_j) \tag{29}$$

V^0_j being the spaces defined in section 2.2.2. By proposition 2.3, this discrete space $\vec{\mathbf{V}}_j$ preserves the divergence-free condition, as stated by Jouini-Lemarié-Rieusset[18]:

$$\mathbf{u} \in (L^2(\Omega))^2,\ \ \mathbf{div}(\mathbf{u}) = 0\ \Rightarrow\ \mathbf{div}[\vec{\mathbf{P}}_j(\mathbf{u})] = 0 \tag{30}$$

where $\vec{\mathbf{P}}_j$ is the biorthogonal projector on $\vec{\mathbf{V}}_j$:

$$\vec{\mathbf{P}}_j = (\mathcal{P}^1_j \otimes \mathcal{P}^0_j, \mathcal{P}^0_j \otimes \mathcal{P}^1_j) \tag{31}$$

In the same way, we now introduce anisotropic divergence-free wavelets and wavelet spaces:

Definition 2.9
The anisotropic divergence-free wavelets and wavelet spaces are given by:

$$\Psi^{div,1}_{\mathbf{j},\mathbf{k}} := \mathbf{curl}[\varphi^D_{j_{min},k_1} \otimes \psi^D_{j_2,k_2}] \quad and \quad \mathbf{W}^{div,1}_{\mathbf{j}} = span\{\Psi^{div,1}_{\mathbf{j},\mathbf{k}}\},\ j_2 \geq j_{min}$$

$$\Psi^{div,2}_{\mathbf{j},\mathbf{k}} := \mathbf{curl}[\psi^D_{j_1,k_1} \otimes \varphi^D_{j_{min},k_2}] \quad and \quad \mathbf{W}^{div,2}_{\mathbf{j}} = span\{\Psi^{div,2}_{\mathbf{j},\mathbf{k}}\},\ j_1 \geq j_{min}$$

$$\Psi^{div,3}_{\mathbf{j},\mathbf{k}} := \mathbf{curl}[\psi^D_{j_1,k_1} \otimes \psi^D_{j_2,k_2}] \quad and \quad \mathbf{W}^{div,3}_{\mathbf{j}} = span\{\Psi^{div,3}_{\mathbf{j},\mathbf{k}}\},\ j_1, j_2 \geq j_{min}$$

The following proposition proves that $(\mathbf{V}^{div}_j)_{j \geq j_{min}}$ is a multiresolution analysis of $\mathcal{H}_{div}(\Omega)$.

Proposition 2.7
The divergence-free scaling functions spaces $\mathbf{V}^{div}_j$ and wavelet spaces $\mathbf{W}^{div,\varepsilon}_{\mathbf{j}}$ for $\varepsilon = 1, 2, 3$, satisfy:
(i) $\mathbf{V}^{div}_{j_{min}} \subset \cdots \subset \mathbf{V}^{div}_j \subset \mathbf{V}^{div}_{j+1} \subset \cdots \subset \mathcal{H}_{div}(\Omega)$ *and* $\overline{\cup \mathbf{V}^{div}_j} = \mathcal{H}_{div}(\Omega)$.
(ii) $\mathbf{V}^{div}_j = \mathbf{V}^{div}_{j_{min}} \bigoplus_{j_{min} \leq j_1, j_2 \leq j-1} (\oplus_{\varepsilon=1,2,3} \mathbf{W}^{div,\varepsilon}_{\mathbf{j}})$.
(iii) For all $\mathbf{j}$ *and* $\varepsilon = 1, 2, 3$, $\{\Psi^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}\}$ *is a Riesz basis of* $\mathbf{W}^{div,\varepsilon}_{\mathbf{j}}$.

Then each vector function $\mathbf{u}$ of $\mathcal{H}_{div}(\Omega)$ has an unique decomposition into the basis $\{\Phi^{div}_{j_{min},\mathbf{k}}, \Psi^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}\}_{j_1,j_2\geq j_{min};\varepsilon=1,2,3}$:

$$\mathbf{u} = \sum_k c^{div}_{j_{min},\mathbf{k}} \Phi^{div}_{j_{min},\mathbf{k}} + \sum_{\mathbf{j},\mathbf{k}} \sum_{\varepsilon=1,2,3} d^{div,\varepsilon}_{\mathbf{j},k} \Psi^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}$$

with the norm-equivalence:

$$\|\mathbf{u}\|_{L^2} \sim \sum_{\mathbf{k}} |c^{div}_{j_{min},\mathbf{k}}|^2 + \sum_{\mathbf{j},\mathbf{k}} \sum_{\varepsilon=1,2,3} |d^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}|^2$$

Proof 2.6 (i) *Let $\vec{\mathbf{V}}_j$ be the spaces defined in* (29)*. Since the spaces $\mathcal{H}_{div}(\Omega)\cap\vec{\mathbf{V}}_j$ provide a multiresolution analysis of $\mathcal{H}_{div}(\Omega)$[18], point (i) is reduced to prove that: $\mathbf{V}^{div}_j = \mathcal{H}_{div}(\Omega) \cap \vec{\mathbf{V}}_j$.*
According to proposition 2.2, we have $\mathbf{V}^{div}_j \subset \vec{\mathbf{V}}_j$ and $\mathbf{V}^{div}_j \subset \mathcal{H}_{div}(\Omega)$ by construction.
Conversely, let $\mathbf{u} \in \mathcal{H}_{div}(\Omega) \cap \vec{\mathbf{V}}_j$, and $\vec{\mathbf{P}}_j$ be the biorthogonal projector on $\vec{\mathbf{V}}_j$ defined in (31)*. We are going to prove that $\mathbf{u} \in \mathbf{V}^{div}_j$. On one side, as $\mathbf{u} \in \vec{\mathbf{V}}_j$ we have $\mathbf{u} = \vec{\mathbf{P}}_j(\mathbf{u})$, on the other hand due to $\mathbf{u} \in \mathcal{H}_{div}(\Omega)$ we have $\mathbf{u} = \mathbf{curl}\ (\chi)$ with $\chi \in H^1_0(\Omega)$, and thus:*

$$\mathbf{u} = \vec{\mathbf{P}}_j[\mathbf{curl}\ (\chi)]$$

Since the spaces $(V^D_j \otimes V^D_j)_{j\geq j_{min}}$ form a MRA of $H^1_0(\Omega)$, we can decompose χ as:

$$\chi = \mathbf{P}^D_j(\chi) + \sum_{j_1,j_2\geq j} \left(\mathbf{Q}^D_1(\chi) + \mathbf{Q}^D_2(\chi) + \mathbf{Q}^D_3(\chi)\right)$$

where

$$\mathbf{P}^D_j(\chi) = \sum_{\mathbf{k}} c_{\mathbf{k}}\ \varphi^D_{j,k_1} \otimes \varphi^D_{j,k_2}, \quad \mathbf{Q}^D_2(\chi) = \sum_{j_1\geq j,\mathbf{k}} d^2_{j_1,\mathbf{k}}\ \psi^D_{j_1,k_1} \otimes \varphi^D_{j,k_2}$$
$$\mathbf{Q}^D_1(\chi) = \sum_{j_2\geq j,\mathbf{k}} d^1_{j_2,\mathbf{k}}\ \varphi^D_{j,k_1} \otimes \psi^D_{j_2,k_2}, \quad \mathbf{Q}^D_3(\chi) = \sum_{j_1,j_2\geq j,\mathbf{k}} d^3_{\mathbf{j},\mathbf{k}}\ \psi^D_{j_1,k_1} \otimes \psi^D_{j_2,k_2}$$

are the biorthogonal projectors on respectively $V^D_j \otimes V^D_j$, $W^D_{j_1} \otimes V^D_j$, $V^D_j \otimes W^D_{j_2}$ and $W^D_{j_1} \otimes W^D_{j_2}$. Proposition 2.2 implies that:

$$\mathbf{curl}\ [\varphi^D_{j,k_1} \otimes \psi^D_{j_2,k_2}] \in (V^D_j \otimes W^0_{j_2}) \times (V^0_j \otimes W^D_{j_2})$$

hence:

$$\vec{\mathbf{P}}_j(\mathbf{curl}\ [\varphi^D_{j,k_1} \otimes \psi^D_{j_2,k_2}]) = 0$$

and same for $\vec{\mathbf{P}}_j(\mathbf{curl}\ [\psi^D_{j_1,k_1} \otimes \varphi^D_{j,k_2}])$ and $\vec{\mathbf{P}}_j(\mathbf{curl}\ [\psi^D_{j_1,k_1} \otimes \psi^D_{j_2,k_2}])$. This leads to:

$$\vec{\mathbf{P}}_j(\mathbf{curl}\ (\chi)) = \vec{\mathbf{P}}_j(\mathbf{curl}\ [\mathbf{P}^D_j(\chi)]) = \mathbf{curl}\ [\mathbf{P}^D_j(\chi)]$$

By construction we have $\mathbf{curl}\ [\mathbf{P}^D_j(\chi)] \in \mathbf{V}^{div}_j$, which implies $\mathbf{u} \in \mathbf{V}^{div}_j$ and then completes the proof: $\mathbf{V}^{div}_j = \mathcal{H}_{div}(\Omega) \cap \vec{\mathbf{V}}_j$.
(ii) *The spaces V^D_j are a multiresolution analysis of $H^1_0(0,1)$, and we can write:*

$$V^D_j \otimes V^D_j = (V^D_{j_{min}} \bigoplus_{j_1=j_{min}}^{j-1} W^D_{j_1}) \otimes (V^D_{j_{min}} \bigoplus_{j_2=j_{min}}^{j-1} W^D_{j_2})$$

By definition of $\mathbf{V}_j^{div}$, *we obtain:*

$$\mathbf{V}_j^{div} = \mathbf{curl}\left[(V_{j_{min}}^D \otimes V_{j_{min}}^D) \bigoplus_{j_{min} \leq j_1, j_2 \leq j-1} [(V_{j_{min}}^D \otimes W_{j_2}^D) \oplus (W_{j_1}^D \otimes V_{j_{min}}^D) \oplus (W_{j_1}^D \otimes W_{j_2}^D)]\right]$$

which is exactly $\mathbf{V}_j^{div} = \mathbf{V}_{j_{min}}^{div} \bigoplus \left[\bigoplus_{j_{min} \leq j_1, j_2 \leq j-1} \left(\oplus_{\varepsilon=1,2,3} \mathbf{W}_{\mathbf{j}}^{div,\varepsilon}\right)\right]$. *(iii) Following [20, 8], this point is a consequence of the proposition 2.8 below.*

We now introduce the biorthogonal divergence-free scaling functions and wavelets. Let:

$$\tilde{\Phi}_{j,\mathbf{k}}^{div} := \left| \begin{array}{l} \tilde{\varphi}_{j,k_1}^D \otimes \tilde{\gamma}_{j,k_2} \\ -\tilde{\gamma}_{j,k_1} \otimes \tilde{\varphi}_{j,k_2}^D \end{array} \right. , \quad \tilde{\Psi}_{\mathbf{j},\mathbf{k}}^{div,1} := \left| \begin{array}{l} 2^{j_2} \tilde{\varphi}_{j_{min},k_1}^D \otimes \tilde{\psi}_{j_2,k_2}^0 \\ -\tilde{\gamma}_{j_{min},k_1} \otimes \tilde{\psi}_{j_2,k_2}^D \end{array} \right. \tag{32}$$

$$\tilde{\Psi}_{\mathbf{j},\mathbf{k}}^{div,2} := \left| \begin{array}{l} \tilde{\psi}_{j_1,k_1}^D \otimes \tilde{\gamma}_{j_{min},k_2} \\ -2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \otimes \tilde{\varphi}_{j_{min},k_2}^D \end{array} \right. , \quad \tilde{\Psi}_{\mathbf{j},\mathbf{k}}^{div,3} := \left| \begin{array}{l} 2^{j_2} \tilde{\psi}_{j_1,k_1}^D \otimes \tilde{\psi}_{j_2,k_2}^0 \\ -2^{j_1} \tilde{\psi}_{j_1,k_1}^0 \otimes \tilde{\psi}_{j_2,k_2}^D \end{array} \right. \tag{33}$$

where: $\tilde{\gamma}_{j,k} = -\int_0^x \tilde{\varphi}_{j,k}^D$. Remark that $\tilde{\gamma}_{j,k}(0) = \tilde{\gamma}_{j,k}(1)$ since $\tilde{\varphi}_{j,k}^D \in H_0^1(0,1)$.

Proposition 2.8
For a fixed $j \geq j_{min}$, *the normalized families*
$\left\{\frac{1}{\sqrt{2}}\Phi_{j,\mathbf{k}}^{div}, \frac{1}{\sqrt{4^{j_2}+1}}\Psi_{\mathbf{j},\mathbf{k}}^{div,1}, \frac{1}{\sqrt{4^{j_1}+1}}\Psi_{\mathbf{j},\mathbf{k}}^{div,2}, \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\Psi_{\mathbf{j},\mathbf{k}}^{div,3} \ ; \ j_1, j_2 \geq j \ , \ \mathbf{k}\right\}$ *and* $\left\{\frac{1}{\sqrt{2}}\tilde{\Phi}_{j,\mathbf{k}}^{div}, \frac{1}{\sqrt{4^{j_2}+1}}\tilde{\Psi}_{\mathbf{j},\mathbf{k}}^{div,1}, \frac{1}{\sqrt{4^{j_1}+1}}\tilde{\Psi}_{\mathbf{j},\mathbf{k}}^{div,2}, \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\tilde{\Psi}_{\mathbf{j},}^{di}\right.$
are biorthogonal in $(L^2(\Omega))^2$, *then they form Riesz sequences of* $(L^2(\Omega))^2$.

Remark 5
Contrarily the usual definitions of MRAs, the (L^2-normalized) divergence-free scaling functions $(\Phi_{j,\mathbf{k}}^{div})_{\mathbf{k}}$ *don't form a Riesz basis of the space* $\mathbf{V}_j^{div}$, *since they do not verify:*

$$\|\sum_{\mathbf{k}} c_{j,\mathbf{k}}^{div} \ \Phi_{j,\mathbf{k}}^{div}\|_{(L^2(\Omega))^2}^2 \sim \sum_{\mathbf{k}} |c_{j,\mathbf{k}}^{div}|^2$$

for all $(c_{j,\mathbf{k}}^{div}) \in \ell^2$ *and independently of* j. *A counterexample is given by:*

$$\mathbf{u} = \sum_{k_\flat \leq k_1, k_2 \leq 2^j - k_\sharp} \Phi_{j,k_1,k_2}^{div},$$

which satisfy: $\sum_{\mathbf{k}} |c_{j,\mathbf{k}}^{div}|^2 = (2^j - k_\sharp - k_\flat + 1)^2 \sim 2^{2j}$.
On the other hand:

$$\mathbf{u} \ = \ \sum_{k_\flat \leq k_1, k_2 \leq 2^j - k_\sharp} \frac{2^{-j}}{\sqrt{2}} \left| \begin{array}{l} \varphi_{j,k_1}^D \otimes (\varphi_{j,k_2}^D)' \\ -(\varphi_{j,k_1}^D)' \otimes \varphi_{j,k_2}^D \end{array} \right.$$

$$\sum_{k_\flat \leq k_2 \leq 2^j - k_\sharp} (\varphi_{j,k_2}^D)' = \sum_{k_\flat \leq k_2 \leq 2^j - k_\sharp} 2^j (\varphi_{j,k_2}^0 - \varphi_{j,k_2+1}^0) = 2^j (\varphi_{j,k_\flat}^0 - \varphi_{j,2^j - k_\sharp + 1}^0)$$

Let $h(x) = \sum_{k_\flat \leq k \leq 2^j - k_\sharp} \varphi_{j,k}^D(x) \sim 2^{j/2} \chi_{[\delta_\flat, \delta_\sharp]}$, $\forall x \in]0,1[$
and $\|\mathbf{u}\|_{(L^2(\Omega))^2}^2 = 2^{-2j} \left(\int_0^1 h^2\right) \left(\int_0^1 2^{2j} (\varphi_{j,k_\flat}^0 - \varphi_{j,2^j - k_\sharp + 1}^0)^2\right) \sim 2^j$

2.4 Biorthogonal MRA of $\mathcal{H}_{div}^{\perp}(\Omega)$

In this section, the curl-free function space $\mathcal{H}_{curl}(\Omega)$ that we consider is the following:

$$\mathcal{H}_{curl}(\Omega) = \{\mathbf{u} = \nabla\ q :\ \ q \in H_0^1(\Omega)\}$$

This space is a proper subspace of $\mathcal{H}_{div}^{\perp}(\Omega)$:

$$\mathcal{H}_{div}^{\perp}(\Omega) = \mathcal{H}_{curl}(\Omega) \oplus \mathcal{H}^{\Delta}(\Omega),\ \text{ with }\ \mathcal{H}^{\Delta}(\Omega) = \{\nabla\ q :\ \ q \in H^1(\Omega)\ \underline{\text{and}}\ \Delta\ q = 0\}$$

To construct a multiresolution analysis of $\mathcal{H}_{curl}(\Omega)$, it suffices to consider the gradient of a MRA of $H_0^1(\Omega)$. With the same notations as in previous section 2.3, curl-free scaling functions spaces are defined by:

Definition 2.10
For $j \geq j_{min}$, a curl-free scaling function space $\mathbf{V}_j^{\nabla}$ is defined by:

$$\mathbf{V}_j^{\nabla} = \nabla(V_j^D \otimes V_j^D) = span\{\Phi_{j,\mathbf{k}}^{\nabla}\} \tag{34}$$

where the curl-free scaling functions are given by:

$$\Phi_{j,\mathbf{k}}^{\nabla} := \nabla[\varphi_{j,k_1}^D \otimes \varphi_{j,k_2}^D],\ j \geq j_{min} \tag{35}$$

The spaces $(\mathbf{V}_j^{\nabla})$ constitute an increasing sequence of subspaces of $(L^2(\Omega))^2$, of dimension: $\dim(\mathbf{V}_j^{\nabla}) = \dim(V_j^D \otimes V_j^D) = (2^j - k_\sharp - k_\flat + 2r - 1)^2$.

Let $\vec{\mathbf{V}}_j^*$ be the standard multiresolution analysis of $(L^2(\Omega))^2$ defined by:

$$\vec{\mathbf{V}}_j^* = (V_j^0 \otimes V_j^1) \times (V_j^1 \otimes V_j^0) \tag{36}$$

By proposition 2.3, the spaces $\mathbf{V}_j^{\nabla}$ are contained in $\vec{\mathbf{V}}_j^*$. We now define the corresponding irrotational wavelets.

Definition 2.11
The anisotropic curl-free wavelets and wavelet spaces are defined by:

$$\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,1} := \nabla[\varphi_{j_{min},k_1}^D \otimes \psi_{j_2,k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{\nabla,1} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,1}\},\ j_2 \geq j_{min}$$

$$\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,2} := \nabla[\psi_{j_1,k_1}^D \otimes \varphi_{j_{min},k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{\nabla,2} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,2}\},\ j_1 \geq j_{min}$$

$$\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,3} := \nabla[\psi_{j_1,k_1}^D \otimes \psi_{j_2,k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{\nabla,3} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,3}\},\ j_1, j_2 \geq j_{min}$$

The following proposition holds:

Proposition 2.9
The spaces $\mathbf{V}_j^{\nabla}$ and $\mathbf{W}_{\mathbf{j}}^{\nabla,\varepsilon}$ for $\epsilon = 1,2,3$ verify:
(i) $\mathbf{V}_{j_{min}}^{\nabla} \subset \cdots \subset \mathbf{V}_j^{\nabla} \subset \mathbf{V}_{j+1}^{\nabla} \subset \cdots \subset \mathcal{H}_{curl}(\Omega)$ *and* $\overline{\cup \mathbf{V}_j^{\nabla}} = \mathcal{H}_{curl}(\Omega)$
(ii) $\mathbf{V}_j^{\nabla} = \mathbf{V}_{j_{min}}^{\nabla} \bigoplus_{j_{min} \leq j_1, j_2 \leq j-1} (\oplus_{\varepsilon=1,2,3} \mathbf{W}_{\mathbf{j}}^{\nabla,\varepsilon})$
(iii) For all $\mathbf{j}$ *and* $\varepsilon = 1,2,3$, $\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,\varepsilon}\}$ *is a Riesz basis of* $\mathbf{W}_{\mathbf{j}}^{\nabla,\varepsilon}$.

The proof uses the same arguments as for proposition 2.7. In addition, these construction of curl-free scaling functions and wavelets can readily be extended to higher dimensions, for more details see [22, 23].

Example 2

3 Fast divergence-free wavelet transform

We describe in this section the practical computation of divergence-free scaling function and wavelet coefficients of a vector field $\mathbf{u} \in \mathbf{V}_j^{div}$. We use the same notations as in previous sections. The starting point is the decomposition of $\mathbf{u}$ in the MRA of $(L^2(\Omega))^2$ provided by $\vec{\mathbf{V}}_j$:

$$\vec{\mathbf{V}}_j = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1)$$

On the scaling functions basis of $\vec{\mathbf{V}}_j$, the vector field $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ can be written as:

$$\mathbf{u}_1 = \sum_{\mathbf{k}} c^1_{j,\mathbf{k}}\ \varphi^1_{j,k_1} \otimes \varphi^0_{j,k_2} \quad \text{and} \quad \mathbf{u}_2 = \sum_{\mathbf{k}} c^2_{j,\mathbf{k}}\ \varphi^0_{j,k_1} \otimes \varphi^1_{j,k_2} \tag{37}$$

The computation of divergence-free coefficients will use a change of bases between $(\frac{d}{dx}\varphi^1_{j,k})$ and $(\varphi^0_{j,k})$. This needs to construct the matrices L_j^0 and L_j^1 introduced in (21) and (22), which we recall the definition:

$$\frac{d}{dx}\varphi^1_{j,k} = \sum_{n=1}^{\dim(V_j^0)} (L_j^1)_{k,n}\ \varphi^0_{j,n} \quad \text{and} \quad -\int_0^x \varphi^0_{j,n} = \sum_{k=1}^{\dim(V_j^1)} (L_j^0)_{n,k}\ \varphi^1_{j,k}$$

So we first give a more precise result on the computation of the elements of these matrices.

Proposition 3.1
Let k_j^* denoted the dimension of V_j^1. The only nonzero elements of matrices L_j^0 and L_j^1 correspond to:
(i) For edge scaling functions and for $2 \le k \le r$, we have:

$$(L_j^1)_{1,r} = -1,\ (L_j^1)_{k,k-1} = 1,\ (L_j^1)_{k,r} = -\tilde{p}^1_{k-1}(k_\flat - 1)$$

$$(L_j^1)_{k_j^*,k_j^*-r} = 1,\ (L_j^1)_{k_j^*-k+1,k_j^*-k+1} = -1,\ (L_j^1)_{k_j^*-k+1,k_j^*-r} = \tilde{p}^1_{k-1}(2^j - k_\sharp + 1)$$

and

$$(L_j^0)_{r,1} = 1,\ (L_j^0)_{k-1,k} = -1,\ (L_j^0)_{k-1,1} = -\tilde{p}^1_{k-1}(k_\flat - 1)$$

$$(L_j^0)_{k_j^*-r,k_j^*} = -1,\ (L_j^0)_{k_j^*-k+1,k_j^*-k+1} = 1,\ (L_j^0)_{k_j^*-k+1,k_j^*} = -\tilde{p}^1_{k-1}(2^j - k_\sharp + 1)$$

(ii) For interior scaling functions and for $r+1 \le k \le k_j^* - r$, we have:

$$(L_j^1)_{k,k-1} = 1, \quad (L_j^1)_{k,k} = -1$$

and

$$(L_j^0)_{k,m} = -1, \quad (L_j^0)_{k,k_j^*} = -1, \quad k \le m \le k_j^* - r$$

Proof 3.1
To obtain the relation on L_j^1, it suffices to use proposition 2.2 written for $j \geq j_{min}$. To obtain the relation on L_j^0, as it is still true by differentiation we get:

$$-\Phi_{j,\ell}^{0,\flat} = \sum_{k=0}^{r-1} (L_j^0)_{\ell,k} \; (\Phi_{j,k}^{1,\flat})' \quad \textit{and} \quad -\Phi_{j,\ell}^{0,\sharp} = \sum_{k=0}^{r-1} (L_j^0)_{k_j^*+\ell,k_j^*+k} \; (\Phi_{j,k}^{1,\sharp})'$$

Then using the proposition 2.2 again, we have:

$$\Phi_{j,\ell}^{0,\flat} = 2^{-j}[(\Phi_{\ell+1}^{1,\flat})' + \tilde{p}_{\ell+1}^1(k_\flat - 1) \; (\Phi_{j,0}^{1,\flat})']$$

and

$$\Phi_{j,\ell}^{0,\sharp}(1-.) = 2^{-j}[(\Phi_{\ell+1}^{1,\sharp}(1-.))' - \tilde{p}_{\ell+1}^1(2^j - k_\sharp + 1) \; (\Phi_{j,0}^{1,\sharp}(1-.))']$$

because:

$$\varphi_{j,k_\flat}^0 = -2^{-j}(\Phi_{j,0}^{1,\flat})' \quad \textit{and} \quad \varphi_{j,2^j-k_\sharp+1}^0 = 2^{-j}(\Phi_{j,0}^{1,\sharp}(1-.))'$$

Similarly for interior scaling functions, by proposition 2.2 we get:

$$\varphi_{j,k}^0 = 2^{-j}(\varphi_{j,k}^1)' + \varphi_{j,k+1}^0$$

and recursively for $k_\flat + 1 \leq k \leq 2^j - k_\sharp$ we deduce that $\varphi_{j,k}^0$ satisfy:

$$\begin{aligned} \varphi_{j,k}^0 &= 2^{-j}(\varphi_{j,k}^1)' + \varphi_{j,k+1}^0 = 2^{-j}[(\varphi_{j,k}^1)' + \cdots + (\varphi_{j,2^j-k_{max}}^1)'] + \varphi_{j,2^j-k_{max}+1}^0 \\ &= 2^{-j}[(\varphi_{j,k}^1)' + \cdots + (\varphi_{j,2^j-k_{max}}^1)' + (\Phi_{j,0}^{1,\sharp}(1-.))'] \end{aligned}$$

This completes the proof.

Now, using matrices L_j^0 and L_j^1 we can rewrite the components of $\mathbf{u}$ as follows:

$$\mathbf{u}_1 = \sum_{\mathbf{k}} c_{j,\mathbf{k}}^1 \; \varphi_{j,k_1}^1 \otimes \varphi_{j,k_2}^0 = -\sum_{\mathbf{k}} [(c_{j,\mathbf{k}}^1) L_j^0]_{\mathbf{k}} \; \varphi_{j,k_1}^1 \otimes (\varphi_{j,k_2}^1)' \tag{38}$$

and

$$\mathbf{u}_2 = \sum_{\mathbf{k}} c_{j,\mathbf{k}}^2 \; \varphi_{j,k_1}^0 \otimes \varphi_{j,k_2}^1 = -\sum_{\mathbf{k}} [L_j^{0\;T}(c_{j,\mathbf{k}}^2)]_{\mathbf{k}} \; (\varphi_{j,k_1}^1)' \otimes \varphi_{j,k_2}^1 \tag{39}$$

If $\mathbf{u} \in \mathbf{V}_j^{div}$, it can be uniquely written as:

$$\mathbf{u} = \sum_{\mathbf{k}} c_{\mathbf{k}}^{div} \Phi_{j,\mathbf{k}}^{div} \tag{40}$$

Therefore, we have the following proposition.

Proposition 3.2
The matrices of coefficients $[c_{j,\mathbf{k}}^1]$ and $[c_{j,\mathbf{k}}^2]$ are linked to the matrix of coefficients $[c_{j,\mathbf{k}}^{div}]$ by:

$$2^{1/2}[c_{j,\mathbf{k}}^{div}] = L_j^{0\;T}[c_{j,\mathbf{k}}^2] - [c_{j,\mathbf{k}}^1] L_j^0 \tag{41}$$

and conversely:

$$[c_{j,\mathbf{k}}^1] = 2^{-1/2}[c_{j,\mathbf{k}}^{div}] L_j^1 \qquad \textit{and} \qquad [c_{j,\mathbf{k}}^1] = -2^{-1/2} L_j^{1\;T}[c_{j,\mathbf{k}}^{div}] \tag{42}$$

Proof 3.2
We assume that the basis of divergence-free scaling functions used is that of the lemma 2.8. With the help of (38) *and* (39)*, by computing directly the inner product we get:*

$$\langle \mathbf{u} / \tilde{\Phi}^{div}_{j,\mathbf{k}} \rangle = 2^{-1/2} \left[L_j^{0\ T} [c^2_{j,\mathbf{k}}] - [c^1_{j,\mathbf{k}}] L_j^0 \right]_{\mathbf{k}}$$

Which proves (41)*. The second relation* (42) *is nothing but the change of basis described by the previous definition of* L_j^1*.*

Now the objective is to compute the divergence-free wavelet coefficients of $\mathbf{u}$:

$$\begin{aligned}\mathbf{u} = & \sum_{\mathbf{k}} c^{div}_{j,\mathbf{k}} \ \Phi^{div}_{j,\mathbf{k}} + \sum_{j_2 \geq j, \mathbf{k}} d^{div,1}_{\mathbf{j},\mathbf{k}} \ \Psi^{div,1}_{\mathbf{j},\mathbf{k}} \\ & + \sum_{j_1 \geq j, \mathbf{k}} d^{div,2}_{\mathbf{j},\mathbf{k}} \ \Psi^{div,2}_{\mathbf{j},\mathbf{k}} + \sum_{j_1, j_2 \geq j, \mathbf{k}} d^{div,3}_{\mathbf{j},\mathbf{k}} \ \Psi^{div,3}_{\mathbf{j},\mathbf{k}}\end{aligned}$$

We start with the standard wavelet decomposition of $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ in $\vec{\mathbf{V}}_j$:

$$\begin{aligned}\mathbf{u}_1 = & \sum_{\mathbf{k}} c^1_{j,\mathbf{k}} \ \varphi^1_{j,k_1} \otimes \varphi^0_{j,k_2} + \sum_{j_2 \geq j, \mathbf{k}} d^{1,1}_{j_2,\mathbf{k}} \ \varphi^1_{j,k_1} \otimes \psi^0_{j_2,k_2} \\ & + \sum_{j_1 \geq j, \mathbf{k}} d^{1,2}_{j_1,\mathbf{k}} \ \psi^1_{j_1,k_1} \otimes \varphi^0_{j,k_2} + \sum_{j_1, j_2 \geq j, \mathbf{k}} d^{1,3}_{\mathbf{j},\mathbf{k}} \ \psi^1_{j_1,k_1} \otimes \psi^0_{j_2,k_2}\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}_2 = & \sum_{\mathbf{k}} c^2_{j,\mathbf{k}} \ \varphi^0_{j,k_1} \otimes \varphi^1_{j,k_2} + \sum_{j_2 \geq j, \mathbf{k}} d^{2,1}_{j_2,\mathbf{k}} \ \varphi^0_{j,k_1} \otimes \psi^1_{j_2,k_2} \\ & + \sum_{j_1 \geq j, \mathbf{k}} d^{2,2}_{j_1,\mathbf{k}} \psi^0_{j_1,k_1} \otimes \varphi^1_{j,k_2} + \sum_{j_1, j_2 \geq j, \mathbf{k}} d^{2,3}_{\mathbf{j},\mathbf{k}} \ \psi^0_{j_1,k_1} \otimes \psi^1_{j_2,k_2}\end{aligned}$$

Which rewrites, using the matrices L_j^0 and L_j^1:

$$\begin{aligned}\mathbf{u}_1 = & -\sum_{\mathbf{k}} [(c^1_{j,\mathbf{k}}) L_j^0]_{\mathbf{k}} \ \varphi^1_{j,k_1} \otimes (\varphi^1_{j,k_2})' + \sum_{j_2 \geq j, \mathbf{k}} d^{1,1}_{j_2,\mathbf{k}} \ \varphi^1_{j,k_1} \otimes \psi^0_{j_2,k_2} \\ & - \sum_{j_1 \geq j, \mathbf{k}} [(d^{1,2}_{j_1,\mathbf{k}}) L_j^0]_{\mathbf{k}} \ \psi^1_{j_1,k_1} \otimes (\varphi^1_{j,k_2})' + \sum_{j_1, j_2 \geq j, \mathbf{k}} d^{1,3}_{\mathbf{j},\mathbf{k}} \ \psi^1_{j_1,k_1} \otimes \psi^0_{j_2,k_2}\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}_2 = & -\sum_{\mathbf{k}} [L_j^{0\ T} (c^2_{j,\mathbf{k}})]_{\mathbf{k}} \ (\varphi^1_{j,k_1})' \otimes \varphi^1_{j,k_2} - \sum_{j_2 \geq j, \mathbf{k}} [L_j^{0\ T} (d^{2,1}_{j_2,\mathbf{k}})]_{\mathbf{k}} \ (\varphi^1_{j,k_1})' \otimes \psi^1_{j_2,k_2} \\ & + \sum_{j_1 \geq j, \mathbf{k}} d^{2,2}_{j_1,\mathbf{k}} \psi^0_{j_1,k_1} \otimes \varphi^1_{j,k_2} + \sum_{j_1, j_2 \geq j, \mathbf{k}} d^{2,3}_{\mathbf{j},\mathbf{k}} \ \psi^0_{j_1,k_1} \otimes \psi^1_{j_2,k_2}\end{aligned}$$

We now prove the following proposition.

Proposition 3.3
For $\epsilon = 1, 2, 3$, the coefficients $[d^{1,\epsilon}_{\mathbf{j},\mathbf{k}}]$ and $[d^{2,\epsilon}_{\mathbf{j},\mathbf{k}}]$ are linked to the divergence-free wavelet coefficients $[d^{div,\epsilon}_{\mathbf{j},\mathbf{k}}]$ by:

$$d^{div,1}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_2}+1}}[2^{j_2}(d^{1,1}_{\mathbf{j},\mathbf{k}}) - L^0_j{}^T(d^{2,1}_{\mathbf{j},\mathbf{k}})]_{\mathbf{j},\mathbf{k}} \tag{43}$$

$$d^{div,2}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+1}}[(d^{1,2}_{\mathbf{j},\mathbf{k}})L^0_j - 2^{j_1}(d^{2,2}_{\mathbf{j},\mathbf{k}})]_{\mathbf{j},\mathbf{k}} \tag{44}$$

$$d^{div,3}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}[2^{j_2}(d^{1,3}_{\mathbf{j},\mathbf{k}}) - 2^{j_1}(d^{2,3}_{\mathbf{j},\mathbf{k}})]_{\mathbf{j},\mathbf{k}} \tag{45}$$

Inversely we have:

$$[d^{1,1}_{\mathbf{j},\mathbf{k}}] = \frac{2^{j_2}}{\sqrt{4^{j_2}+1}}[d^{div,1}_{\mathbf{j},\mathbf{k}}] \quad \text{and} \quad [d^{2,1}_{\mathbf{j},\mathbf{k}}] = -\frac{1}{\sqrt{4^{j_2}+1}}L^1_j{}^T[d^{div,1}_{\mathbf{j},\mathbf{k}}] \tag{46}$$

$$[d^{1,2}_{\mathbf{j},\mathbf{k}}] = \frac{1}{\sqrt{4^{j_1}+1}}[d^{div,2}_{\mathbf{j},\mathbf{k}}]L^1_j \quad \text{and} \quad [d^{2,1}_{\mathbf{j},\mathbf{k}}] = -\frac{2^{j_1}}{\sqrt{4^{j_1}+1}}[d^{div,2}_{\mathbf{j},\mathbf{k}}] \tag{47}$$

$$[d^{1,3}_{\mathbf{j},\mathbf{k}}] = \frac{2^{j_2}}{\sqrt{4^{j_1}+4^{j_2}}}[d^{div,3}_{\mathbf{j},\mathbf{k}}] \quad \text{and} \quad [d^{2,3}_{\mathbf{j},\mathbf{k}}] = -\frac{2^{j_1}}{\sqrt{4^{j_1}+4^{j_2}}}[d^{div,3}_{\mathbf{j},\mathbf{k}}] \tag{48}$$

Proof 3.3
We assume also that the basis of divergence-free wavelet used is that of the lemma 2.8. The formula are obtained by considering the inner products:

$$\langle \mathbf{u}/\tilde{\Psi}^{div,1}_{\mathbf{j},\mathbf{k}}\rangle, \quad \langle \mathbf{u}/\tilde{\Psi}^{div,2}_{\mathbf{j},\mathbf{k}}\rangle \quad \text{and} \quad \langle \mathbf{u}/\tilde{\Psi}^{div,3}_{\mathbf{j},\mathbf{k}}\rangle$$

The algorithm of reconstruction is still a consequence of proposition 2.2.

Example 3
We start with a vector field $\mathbf{u}$ *arising from a numerical simulation of lid driven cavity flow. Then, we compute its divergence-free scaling function and wavelet coefficients using proposition 3.2 and proposition 3.3. The divergence-free wavelets are constructed from the B-Spline generators of Figure 1 with the same parameters. Figure 9 shows the plot of this vector field and corresponding coefficients.*

4 Applications

In this section, we illustrate some practical uses of the divergence-free and curl-free wavelets constructed before. We first show on a numerical example their powerful properties of nonlinear approximation. Then, we present their application on two problems relevant for the numerical simulation of incompressible flows: the Helmholtz decomposition and the Stokes problem, with homogeneous boundary conditions.

4.1 Nonlinear approximation

Divergence-free and curl-free wavelet bases provide nonlinear approximation estimates, governed by the approximation orders of one-dimensional spaces involved in their construction. In this part, we investigate the convergence rate obtained from the N-best terms approximation.

For numerical test, we used the vector field and wavelets of example 3. On Figure 10, we plot the ℓ_2 error on each component through a solution recovered by non-linear approximation on the divergence-free wavelet basis. We get the same result as on the standard multiresolution analysis provided by $\vec{\mathbf{V}}_j$. This is consistent with the theoretical result proved in [22]. Figure 11 show the classical boundary wavelet error phenomena. It is well know that this error does not prevent the convergence of the multi-scale projector of these bases.

4.2 Helmholtz decomposition

The Helmholtz decomposition of a vector field $\mathbf{u}$ of $(L^2(\Omega))^2$, is a unique decomposition of $\mathbf{u}$ of the form:

$$\mathbf{u} = \mathbf{curl}\ (\chi) + \nabla q \tag{49}$$

with $\chi \in H_0^1(\Omega)$ and $q \in H^1(\Omega)$. The objective in this section is to compute an approximation u_j^{div} in $\mathbb{V}_j^{div}$ of the divergence-free part $u^{div} = \mathbf{curl}\ (\chi)$, using the divergence-free bases built in section 2.3. For simplicity, we use the scaling function basis $(\Phi_{j,\mathbf{k}}^{div})$, since the wavelet one can be deduced using one-dimensional fast wavelet transform along each direction.
$\mathbf{u}_j^{div}$ is searched as its decomposition onto divergence-free scaling functions:

$$\mathbf{u}_j^{div} = \sum_{\mathbf{k}} c_{j,\mathbf{k}}^{div}\ \Phi_{j,\mathbf{k}}^{div} \tag{50}$$

By orthogonality of the decomposition (49) in $(L^2(\Omega))^2$, one obtains:

$$\langle \mathbf{u}/\Phi_{j,\mathbf{k}}^{div}\rangle = \langle \mathbf{u}_j^{div}/\Phi_{j,\mathbf{k}}^{div}\rangle \quad \text{thus} \quad \mathbb{M}(c_{j,\mathbf{k}}^{div}) = (\langle \mathbf{u}/\Phi_{j,\mathbf{k}}^{div}\rangle) \tag{51}$$

where $\mathbb{M}$ the Gram matrix of the basis $\{\Phi_{j,\mathbf{k}}^{div}\}$. The computation of the coefficients $(c_{j,\mathbf{k}}^{div})$ is then reduced to the resolution of a linear system of matrix $\mathbb{M}$. This system can be easily inverted, since $\mathbb{M}$ is no more than the stiffness matrix of a standard Laplacian onto the scalar scaling function basis $\{\varphi_{j,k_1}^D \otimes \varphi_{j,k_2}^D\}$. Indeed:

$$\forall\ \psi, \phi \in H_0^1(\Omega); \quad \int_\Omega \nabla\psi \cdot \nabla\phi dx = \int_\Omega \mathbf{curl}(\psi) \cdot \mathbf{curl}(\phi) dx \tag{52}$$

More details on the implementation and resolution of (51) can be found in [23]. Figure 12 presents the ℓ_2-error of convergence of the algorithm, according to the space resolution j, using wavelets of example 3. The exact solution $\mathbf{u}$ corresponds to:

$$\mathbf{u} = \mathbf{curl}\ [\sin(2\pi x)x^2(1-x)^2y^2(1-y)^2] + \nabla\ [\cos(2\pi x)x^2y^2]$$

4.3 Stokes problem

The Stokes problem is a simple test case for the simulation of incompressible flows. In the non-stationary case, and for a velocity $\mathbf{u}$ vanishing at the boundary, it is described by the following equations:

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla\mathbf{p} = f \;\text{ in }\; [0,T]\times\Omega \\ \nabla\cdot\mathbf{u} = 0 \;\text{ in }\; \Omega \\ \mathbf{u} = 0 \;\text{ on }\; \Gamma \\ \mathbf{u}(0,x) = \mathbf{u}_0 \end{cases} \tag{53}$$

where $\mathbf{p}$ is the corresponding pressure.

K. Urban was the first who uses *interior* divergence-free wavelets for the resolution of the stationary case[27]. His method uses a variational method [16] in $\mathcal{H}_{div}(\Omega)$ and thus requires the inversion of the stiffness matrix in the divergence-free wavelet basis.

We propose here to use the Helmholtz decomposition to simplify a classical method of resolution, called the Chorin projection method [6]. Our algorithm to compute $\mathbf{u}^n(x) \approx \mathbf{u}(x, n\delta t)$ is the following:

Starting with initial values $\mathbf{u}^0 = \mathbf{u}(0,x)$, repeat for $1 \le n \le N$

Step 1: Find $\mathbf{a}(x)$ solution of

$$\frac{\mathbf{a}-\mathbf{u}^n}{\delta t} = \nu\Delta\frac{1}{2}(\mathbf{a}+\mathbf{u}^n) + \mathbb{P}(f), \quad x \in\; [0,1]^2 \tag{54}$$

$$\mathbf{a} = 0 \;\text{ on }\; \Gamma \tag{55}$$

Step 2 Find $\mathbf{u}^{n+1}$ solution of

$$\mathbf{u}^{n+1} = \mathbb{P}(\mathbf{a}) \tag{56}$$

where $\mathbb{P}$ is the orthogonal projector from $(L^2(\Omega))^2$ onto $\mathcal{H}_{div}(\Omega)$ computed in practice by the Helmholtz decomposition described in section 4.2.

This method has the advantage of decoupling the resolution of the diffusion term and the incompressibility constraint. Moreover, in more general boundary condition ($\mathbf{u} \neq 0$ on Γ) there is no need of homogenization for the divergence-free basis like in [25, 27], we can incorporate this boundary condition directly in the basis of $\mathcal{H}_{div}(\Omega)$ used in the computation of $\mathbb{P}(\mathbf{a})$.

On Figure 13, we plot the ℓ_2 error on $\mathbf{u}$ and on its gradient $\nabla\mathbf{u}$ in $\vec{\mathbf{V}}_j$. The exact solution $\mathbf{u}$ is taken from [19]:

$$\mathbf{u}(x,y,t) = \mathbf{curl}\;[1000x^2(1-x)^2y^2(1-y)^2] \qquad \nabla\mathbf{p} = x^2 + y^2 - \frac{2}{3} \tag{57}$$

We used the divergence-free wavelets of example 3 to compute the projector $\mathbb{P}$.

5 Conclusion

In this article we have presented a practical construction of divergence-free and irrotational multiresolution analyses and wavelets. Our construction, based on one-dimensional

analyses on the interval allowing the reproduction of polynomials, respects the theoretical framework of the previous work of Jouini and Lemarié-Rieusset [18]. Moreover our construction can incorporate homogeneous boundary conditions in the basis functions, which allows the representation of more physical divergence-free vector functions. This ability is not present, for instance, in the attempt addressed by Stevenson [24].
Associated fast wavelet transforms have been implemented satisfactory, opening new prospects for the realistic simulation of incompressible flows. First attempts have successfully been presented in this article with the Helmholtz decomposition of a vector flow, or with the computation of a Stokes problem solution. Work on more complex problems are underway, such as the direct simulation of turbulence, and this will the subject of a forthcoming paper.

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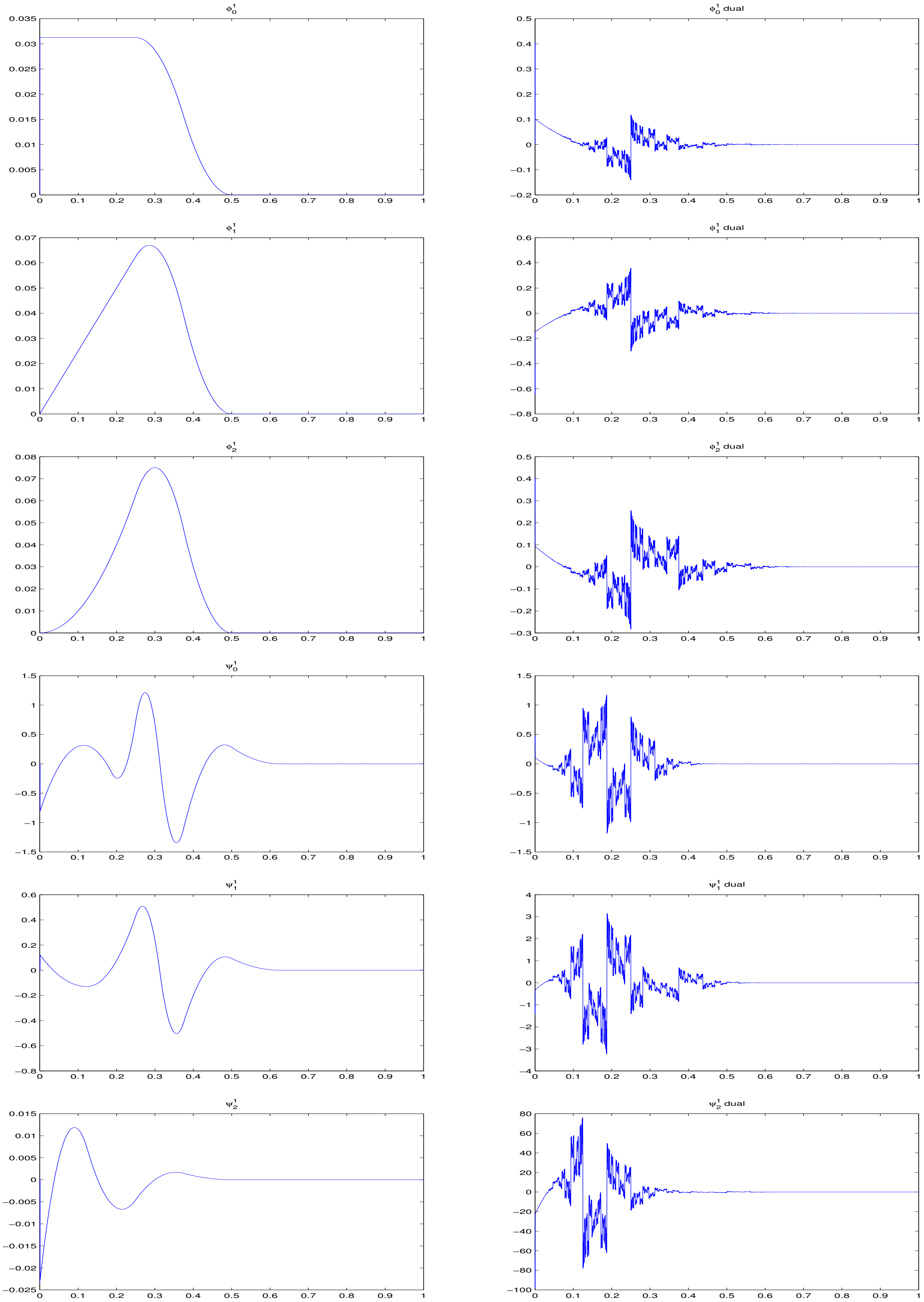

Figure 1: Scaling functions $\Phi_\ell^{1,\flat}$ (three first left) and wavelets $\Psi_\ell^{1,\flat}$ (last three left), their duals scaling functions $\tilde{\Phi}_\ell^{1,\flat}$ (three first right) and wavelets $\tilde{\Psi}_\ell^{1,\flat}$ (last three right).

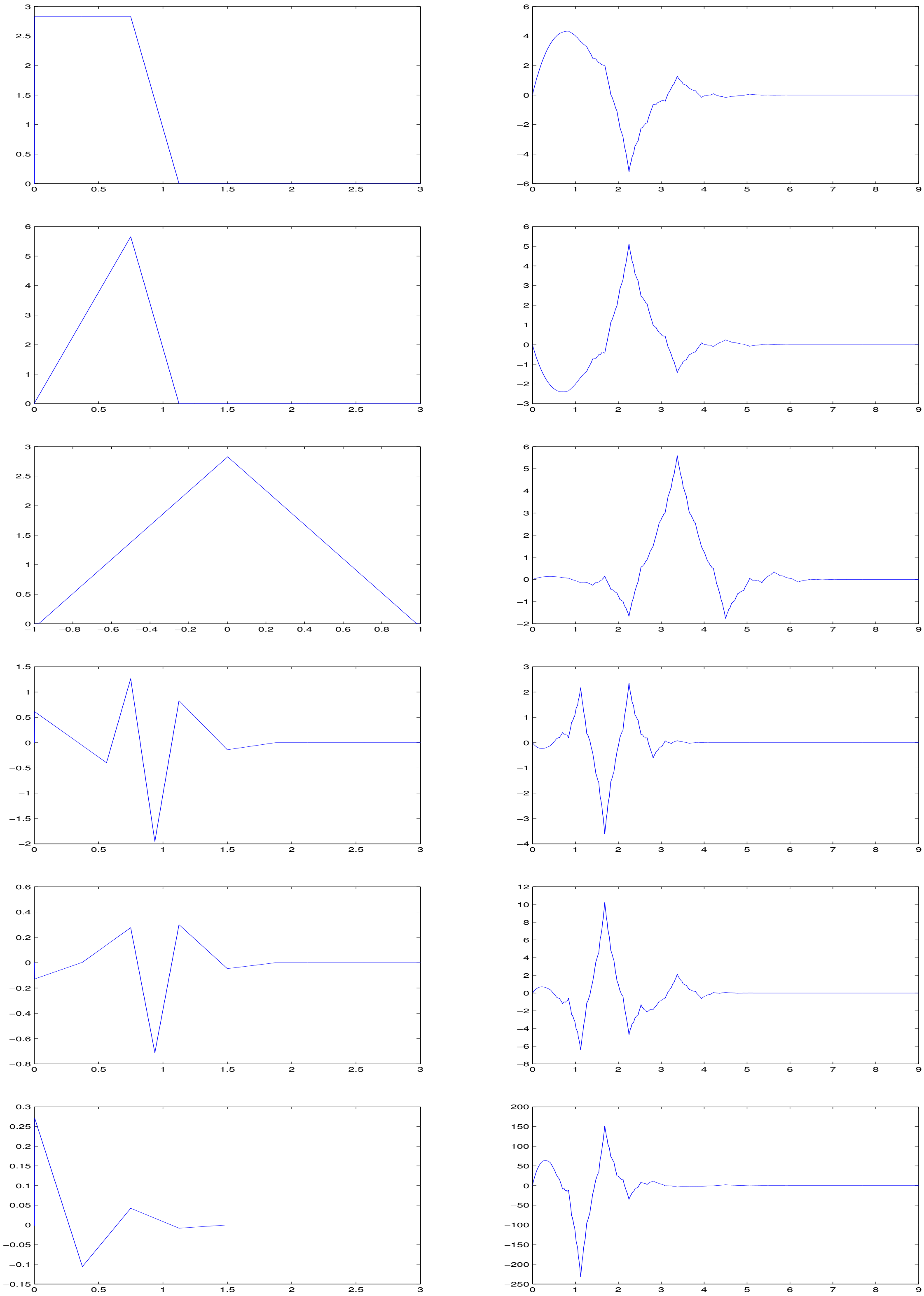

Figure 2: Scaling functions $\Phi_\ell^{0\flat}$ (three first left) and wavelets $\Psi_\ell^{0\flat}$ (last three left), their duals scaling functions $\tilde{\Phi}_\ell^{0\flat}$ (three first right) and wavelets $\tilde{\Psi}_\ell^{0\flat}$ (last three right) provided Dirichlet boundary condition.

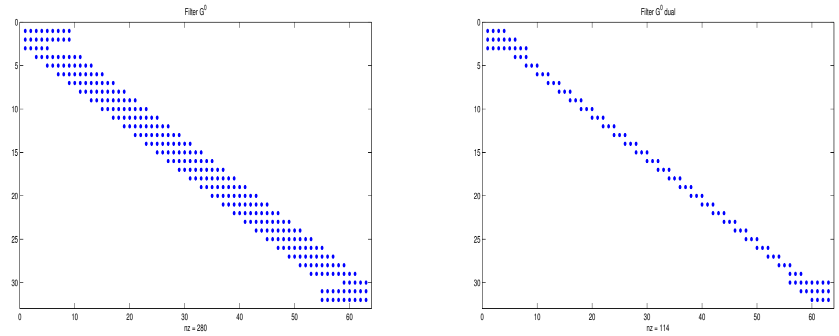

Figure 3: Wavelet filter G_j^0 (left) and wavelet filter $\tilde{G}_j^0$ (right), for $j = 6$.

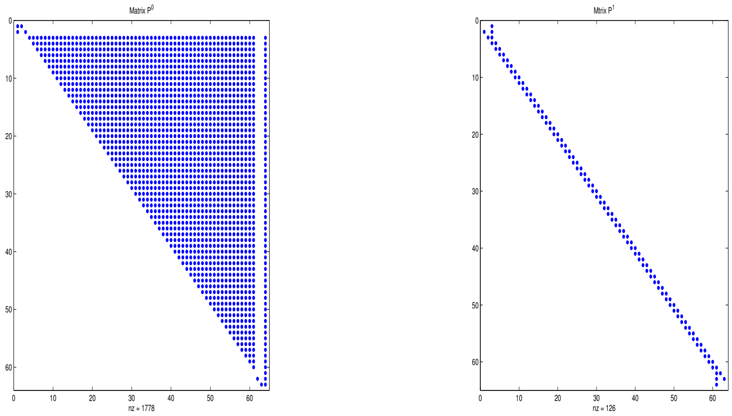

Figure 4: Matrices of change of bases L_j^0 (left) and L_j^1 (right), for $j = 6$.

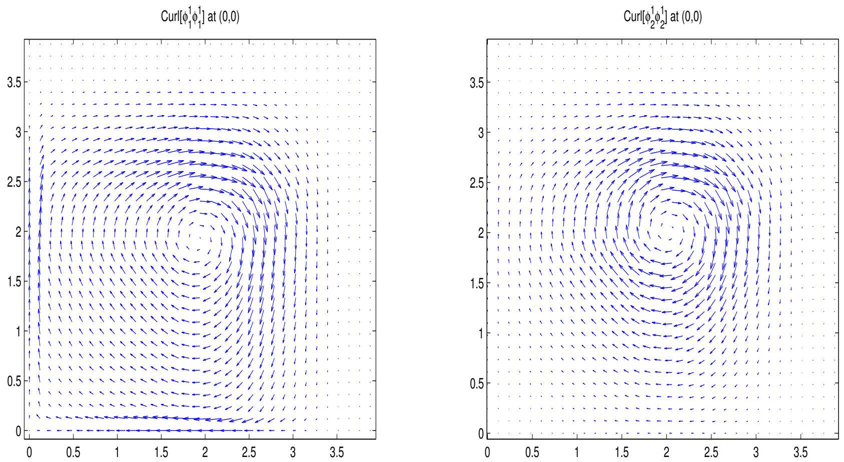

Figure 5: Vector field of $\mathbf{curl}[\Phi_1^{1,\flat} \otimes \Phi_1^{1,\flat}]$ and $\mathbf{curl}[\Phi_2^{1,\flat} \otimes \Phi_2^{1,\flat}]$.

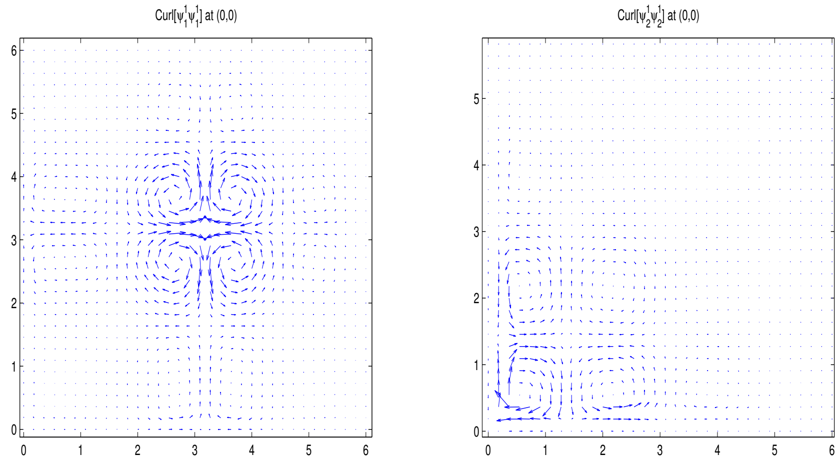

Figure 6: Vector field of $\mathbf{curl}[\Psi_1^{1,\flat} \otimes \Psi_1^{1,\flat}]$ and $\mathbf{curl}[\Psi_2^{1,\flat} \otimes \Psi_2^{1,\flat}]$.

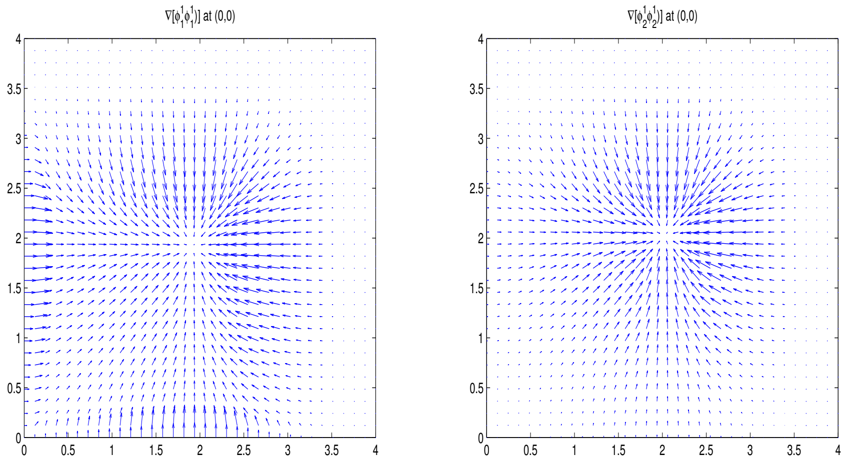

Figure 7: Vector field of $\nabla[\Phi_1^{1,\flat} \otimes \Phi_1^{1,\flat}]$ and $\nabla[\Phi_2^{1,\flat} \otimes \Phi_2^{1,\flat}]$.

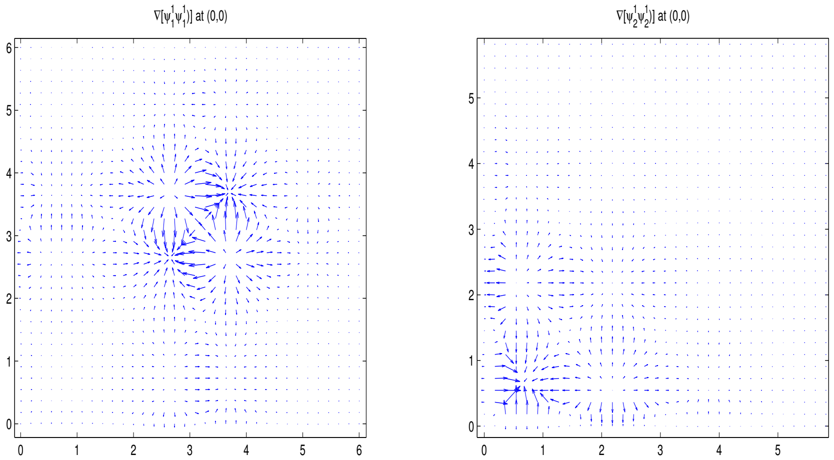

Figure 8: Vector field of $\nabla[\Psi_1^{1,\flat} \otimes \Psi_1^{1,\flat}]$ and $\nabla[\Psi_2^{1,\flat} \otimes \Psi_2^{1,\flat}]$.

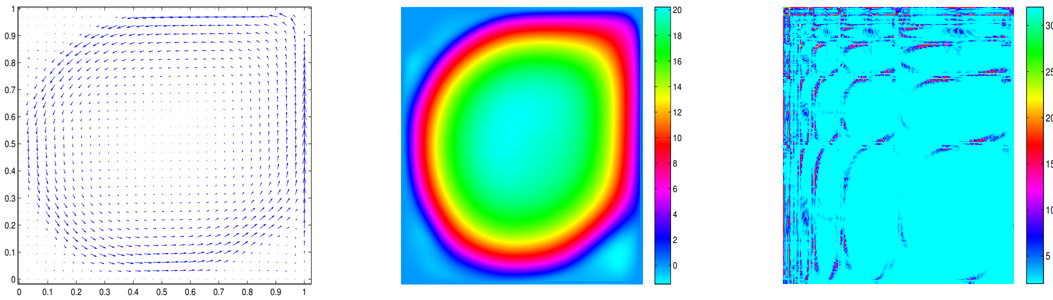

Figure 9: Example of vector field, its divergence-free scaling function coefficients and renormalized wavelet coefficients.

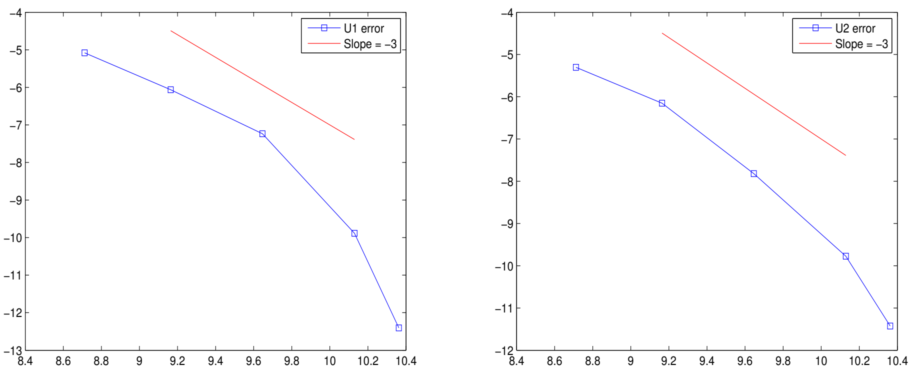

Figure 10: Non linear error approximation: first component $\mathbf{u}_1$ left and second component $\mathbf{u}_2$ right.

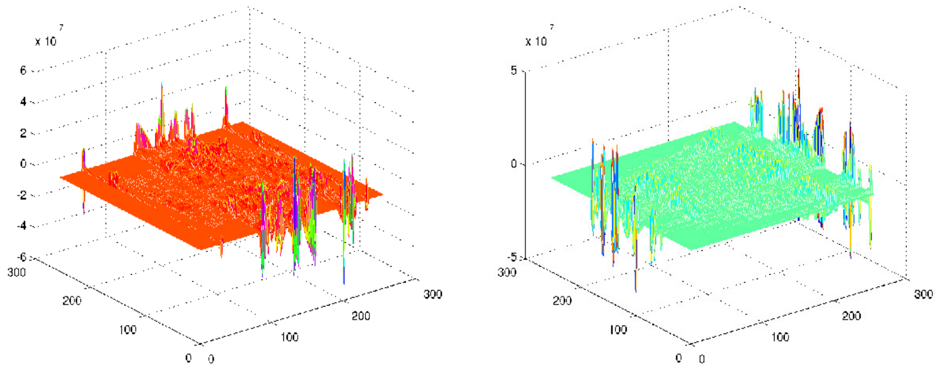

Figure 11: Error on $\mathbf{u}_1$ (left) and $\mathbf{u}_2$ (right) reconstructed from 20% of their divergence-free wavelet coefficients.

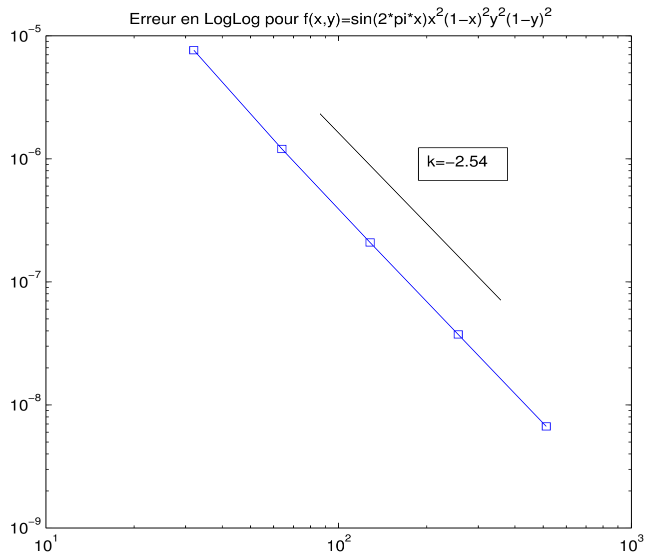

Figure 12: Relative l_2 norm of the projection on this base of $\mathbf{curl}f(x,y)$ with $f(x,y) = \sin(2\pi x)x^2(1-x)^2y^2(1-y)^2$.

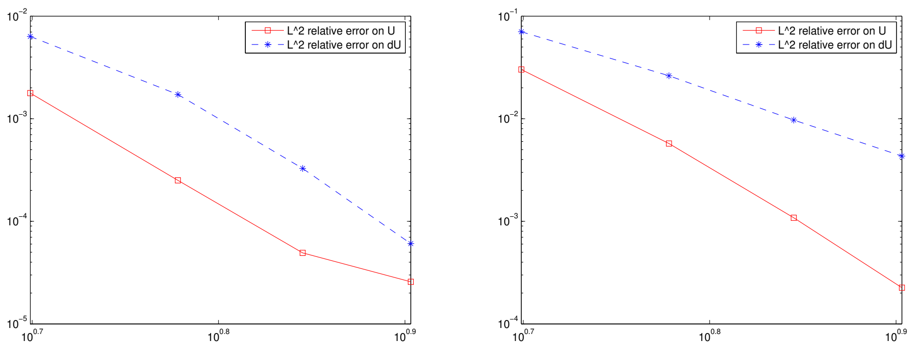

Figure 13: Relative L^2 errors on $\mathbf{u}$ and $\nabla\mathbf{u}$ according to the resolution $\mathbf{j}$, at $t = 10^{-2}$ and $\eta = 10^{-3}$.